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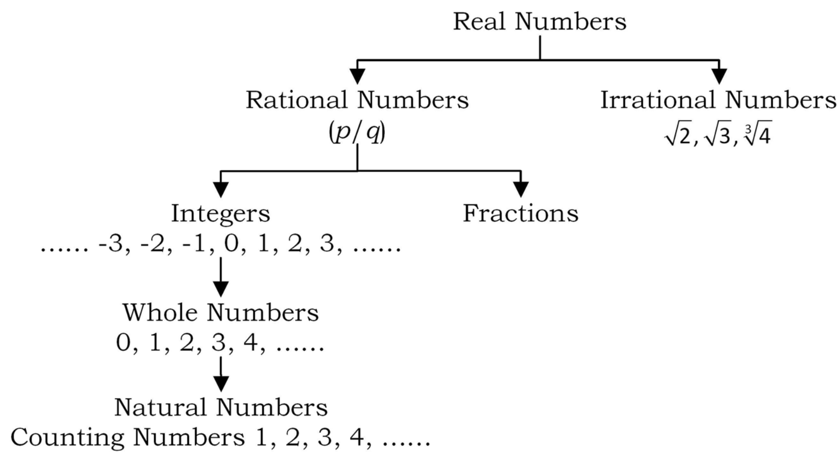
Number System

Number Tree



Notes:

0 is neither positive nor negative.

Positive Integers are 1, 2, 3, ..., i.e. the set of Natural Numbers

Non-negative Integers are 0, 1, 2, 3, ..., i.e. the set of Whole Numbers

Even and Odd

An Even number is a number divisible by 2. Thus Even numbers are represented by $2n$, where n can take values 0, 1, 2, ... Thus Zero is also an Even number.

An Odd number is a number that is not divisible by 2. Thus, Odd numbers are represented by $2n + 1$, where n can take values 0, 1, 2, ...

Addition of numbers:

Even ± Even = Even

Odd ± Odd = Even

Even ± Odd = Even

Multiplication of numbers:

Even × Any number = Even. Hence we have $\text{Even}^{(\text{any number})}$ = Even.

Odd × Odd × Odd × ...... = Odd. Hence we have $\text{Odd}^{(\text{any number})}$ = Odd

E.g. 1: Choose the best option that will fill in the blank: $x^2 + x^3$ is ______________________.

a. always even b. always odd c. may be even or odd depending on x.

If x is even, then x^2 and x^3 are even and the sum of two even numbers is even.

If x is odd, then both, x^2 and x^3, are odd and the sum of two odd numbers will again be even.

Thus, the sum is always even irrespective of x.

E.g. 2: Let x, y, and z be distinct integers. x and y are odd and positive and z is even and positive. Which one of the following statements cannot be true?

a. $(x - z)^2 \times y$ is even b. $(x - z) \times y^2$ is odd c. $(x - z) \times y$ is odd d. $(x - y)^2 \times z$ is even

Evaluating each option turn by turn …

a. $(x - z)$ will be (odd – even) i.e. odd and hence $(x - z)^2$ is also odd.

When this odd number is multiplied with y, another odd number, the result will be odd.

So this option cannot be true. And this is our answer. Even though we need not check further, just for practice …

b & c. As seen above $(x - z)$ will be odd and when it is multiplied with other odd numbers (y^2 or y), the result will be odd. And thus both these options are true.

d. $(x - y)$ will be (odd – odd) i.e. even and when it is multiplied with any number, z in this case, the result will be even. Hence this option is also true.

Divisibility Rules:

Following are the rules to check if a number is divisible by another number.

Rule for 3:

Sum of digits should be divisible by 3.

E.g. the number 34728 is divisible by 3 because 3 + 4 + 7 + 2 + 8 = 24 is divisible by 3.

The number 13069 is not divisible by 3 because 1 + 3 + 0 + 6 + 9 = 19 is not divisible by 3. When the number 13069 is divided by 3 the remainder will be same as that when 19 is divided by 3 i.e. 1

Rule for 4:

The last two digits should be divisible by 4.

If the last two digits is ‘00’, then also the number will be divisible by 4.

E.g. The number 34728 is divisible by 4 because 28 is divisible by 4

The number 13070 is not divisible by 4 because 70 is not divisible by 4. When the number 13070 is divided by 4, the remainder will be same as that when 70 is divided by 4 i.e. 2

Rule for 6:

Check for divisibility by 2 and 3. Only if the number is divisible by both 2 and 3, the number will be divisible by 6.

Rule for 8:

The last three digits should be divisible by 8.

If the last three digits are ‘000’, then also the number will be disable by 8.

E.g. The number 17243632 is divisible by 8 as 632 is divisible by 8.

The number 1430254 is not divisible by 8 as 254 is not divisible by 8. The remainder when 1430254 is divided by 8 will be same as the remainder when 254 is divided by 8 i.e. 6

Rule for 9:

Sum of digits should be divisible by 9.

E.g. The number 14043573 is divisible by 9 because the sum of digits i.e. 27 is divisible by 9.

The number 24736 is not divisible by 9 because the sum of digits is 22 which is not divisible by 9. When the number is divided by 9, the remainder will be same as the remainder when 22 is divided by 9 i.e. 4

Rule for 11:

Add all the alternate digits starting with the digit in units place. Let this sum be U. Add all the remaining alternate digits and let his sum be T. If the difference between U and T is divisible by 11, the number is also divisible.

If the difference between U and T is zero, then also the number will be divisible by 11.

E.g. Consider the number 39061231. Adding alternate digits starting with the unit digit, we get U = 1 + 2 + 6 + 9 = 18. Adding the other set of alternating digits, we get T = 3 + 1 + 0 + 3 = 7. Since U – T = 18 – 7 = 11, which is divisible by 11, the number 39061231 will also be divisible by 11.

E.g. 3: How many distinct values can x assume if 28357x4 is divisible by 8?

Since rule for 8 states that the last three digits should be divisible by 8, $7x4$ should be divisible by 8.

Since we cannot divide $7x$ by as we do not know x, lets start with assuming $x = 0$ and then proceed.

For $x = 0$, the last three digits is 704, which is divisible by 8 and thus x can assume the value 0.

Now as we increment x by 1, we keep adding 10 to the number $7x4$. Since 704 is divisible by 8, on adding 10 or 20 or 30 to the number, we will not get a number divisible by 8 (because 10, 20 or 30 are not divisible by 8). Only on adding 40 or 80 to 704 would we get a number divisible by 8. Thus x can assume values of 0, 4 or 8 i.e. 3 values.

E.g. 4: If the number $18601x57y$ is divisible by 72, find the value of $x + y$.

Since the number is divisible by 72, the number will be divisible by 8 and 9. Rather than starting with rule of 9, which will include both the unknowns x and y, you should start with rule of 8 because this will involve only y.

$57y$ should be divisible by 8. Since 57 divided by 8 leaves a remainder of 1, $1y$ should be divisible y 8 i.e. y can be only 6.

Now, $18601x576$ is divisible by 9 and thus the sum of digits should be divisible by i.e. $34 + x$ should be divisible by 9 i.e. x has to be 2.

Thus $x + y = 6 + 2 = 8$.

E.g. 5: If the number $1735086y4$ is divisible by 11, what is the value that x can assume?

Sum of digits in alternate places is 4 + 6 + 0 + 3 + 1 i.e. 14 and $y + 8 + 5 + 7$ i.e. $20 + y$.

If now one assumes the difference between the sum as zero …

$14 – (20 + y) = 0$ i.e. $y = –6$, which is not possible because y is a digit of the number and cannot be negative.

Thus, the difference between the two sum has to be taken as 11 and we will get ….

$14 – (20 + y) = 0$ i.e. $y = 5$.

Exercise

1. Let x, y and z be distinct integers that are odd and positive. Which of the following statements cannot be true?

   a. $x \times y \times z^2$ is odd
   b. $(x - y)^2 \times z$ is even
   c. $(x + y - z)^2 \times (x + y)$ is even
   d. $(x - y) \times (y + z) \times (x + y - z)$ is odd

2. If a, b, and c are intergers, then (a – b) × (b – c) × (c – a) is

   a. always even
   b. always odd
   c. may be even or odd depending on values.

3. The number 94220p31q is divisible by 88. What is the value of $p + q$?

   a. 7 b. 9 c. 11 d. 13

4. Find the value of x if the number 58215x237 is divisible by 11?

   a. 8 b. 7 c. 6 d. 9

5. If the number 63576x2 is divisible by 8, find the least value that x can take.

   a. 7 b. 3 c. 5 d. 4

6. If the number 86325x6 is divisible by 11, find the least value that x can take.

   a. 4 b. 2 c. 3 d. 6

7. If the number 42573x is divisible by 72, find the value that x can assume.

   a. 4 b. 6 c. 8 d. 7

8. How many different values can x take if the number 2506x8 is divisible by 8?

   a. 1 b. 2 c. 3 d. 4

HCF (or GCD)

The concept of HCF:

Highest Common Factor (or Greatest Common Divisor) for a set of numbers refers to the largest number that can completely divide all the numbers in the given set.

Thus, if we are interested in finding the largest number that can completely divide 72, 108 and 126, we are basically searching for the HCF of {72, 108, 126}.

Numbers that can completely divide a given number n, are factors of the number.

Factors of 72: 1, 2, 3, 4, 6, 8, 9, 12, 18, 24, 36, 72.

These are ALL the numbers that can divide 72. There are no other numbers by which 72 can be completely divisible.

Factors of 108: 1, 2, 3, 4, 6, 9, 12, 18, 27, 36, 54, 108

Factors of 126: 1, 2, 3, 6, 7, 9, 14, 18, 21, 42, 63, 126

The Common Factors i.e. 1, 2, 3, 6, 9 and 18 are the numbers that can completely divide each of the three given numbers viz. 72, 108, 126.

There is no other number that can divide each of the three numbers e.g. both 72 and 108 are divisible by 4, but 126 is not.

Among these Common Factors, the highest is 18. Thus we can safely say that 18 is the largest number that can divide each of 72, 108 and 126. There is no number higher than 18 that can divide the given numbers. So the Highest Common Factor of 72, 108 and 126 is 18.

Thus, say 72 apples, 108 oranges and 126 mangoes had to be divided into smaller groups (without mixing the fruits) such that each group had the same number of fruits. The number of fruits in a group will have to be factor of each of 72, 108 and 126. Hence, the largest possible number of fruits in one group could be 18. A group of more than 18 is not possible. The 72 apples and 108 oranges could be divided into groups of 36 each, but the 126 oranges could not be divided into groups of 36. This example here forms the platform of applications of HCF. So be sure to understand it and the use of HCF completely before proceeding.

Finding the HCF

Approach 1: Factorisation

Factorise each of the given number. The HCF will be the product of the least powers of primes across the given numbers.

$72 = 2^3 \times 3^2$

$108 = 2^2 \times 3^3$

$126 = 2 \times 3^2 \times 7$

Thus, HCF = $2 \times 3^2 = 18$.

Obviously, this method is too text-bookish and should not be resorted to while find the HCF.

Approach 2: Finding the highest power of primes that divides each of the numbers.

If the numbers are very easy, like the above example, 72, 108 and 126, just find the highest power of each primes (2, 3, 5, 7, … taken turn by turn) that can divide each of the given numbers.

E.g. To find the HCF of 72, 108 and 126…

Considering powers of 2, while 72 can be divided by 2, 4 and 8; 108 can be divided by only 2 and 4; 126 only by 2. Thus HCF, being a common factor, should just have one power of 2 (else it would not be a factor of 126)

Considering powers of 3, 72 can be divided by 3 and 9; 108 by 3, 9 and 27; 126 by 3 and 9. Thus HCF should just have second power of 3

5, 7 or higher primes do not divide atleast one of the numbers. So they would not not be present in the HCF.

Thus HCF is 2×3^2.

E.g. Find the HCF of 180, 300 and 420

Checking for divisibility by 2, 4 and 8, we find that each of the number can be divided by 4.

Checking for divisibility by 3 and 9, while 180 is divisible by 9, 300 and 420 are divisible by only 3.

Also all the three numbers can be divided by 5.

No other prime number needs to be checked because 180 can be factorised into only 2, 3 and 5.

Thus HCF = $4 \times 3 \times 5 = 60$.

Approach 3:

Write the given numbers as a product of smaller numbers, anything that comes to your mind first. E.g. Let's say we think of:

$72 = 6 \times 12$ $108 = 12 \times 9$ $126 = 2 \times 63$

Since 63 is just 7×9, we just have to check if the other two numbers can be divided by any of 2, 7 and 9. Obviously, both numbers can be divided by 2 and 9 but not by 7. Thus, we can conclude that 2×9 is the highest factor common to all the three numbers.

E.g. Find the HCF of 180, 300 and 420

$180 = 18 \times 10$ $300 = 30 \times 10$ $420 = 42 \times 10$

Thus 10 is common to all the three numbers and from the rest {18, 30 and 42}, we can find 6 to be the highest common factor.

Thus HCF = $10 \times 6 = 60$.

Application of HCF:

1. Find the largest number that divides the given numbers

E.g. 6: Find the largest number that on dividing 85, 150 and 220 leaves a remainder of 1, 6 and 4 respectively.

The number on dividing 85 leaves a remainder of 1. Thus, had the 1 not be present, the required number would have divided the rest completely i.e. the required number can completely divide $85 – 1 = 84$.

Similarly the number can completely divide 15 0 – 6 = 144 and 220 – 4 = 216.

Thus we are looking for the largest number that completely divides 84, 144 and 216 i.e. HCF of 84, 144 and 216.

Finding the HFC: Starting with $84 = 12 \times 7$, its obvious that 7 does not divide the other numbers. So checking with 12, we find that each of the other numbers is divisible by 12. Thus HCF of 84, 144 and 216 (and the required answer) is 12.

E.g. 7: Find the largest possible three digit number that leaves the same remainder when it divides 145, 277, 673.

Let's assume the number we want to find to be n and the common remainder in each case to be r. Thus we have,

$145 = n \times a + r$

$277 = n \times b + r$

$673 = n \times c + r$

Subtracting the equations we have,

$132 = n \times (b - a)$

$396 = n \times (c - b)$

$528 = n \times (c - a)$

Thus n has to be a factor of 132, 396 and 528. Finding the HCF of these numbers,

$132 = 11 \times 12$. Checking if the other two numbers are divisible by 11 and 12, we find that they are (just apply divisibility rules of 11, 4 and 3).

Thus, the largest value that n can take is 132.

(Answer check: 132 on dividing 145, 277 and 673 leaves a remainder of 13 in each case)

2. Grouping or dividing two or more quantities:

E.g. 8: 84 rose plants, 126 marigold plants and 189 chrysanthemum plants have to be planted in rows such that each row has equal number of plants and each row has plants of a particular variety only. What is the least number of rows needed?

Since (number of plants in each row) × (number of rows) = (total number of plants), the number of plants in each row has to be a factor of each of total number of plants of each variety i.e. a Common Factor of 84, 126 and 189.

Its important to realize the above because only this is going to give a hint that we have to work with factors and find the HCF.

The number of rows will be least when each row has as many plants as possible, i.e. each row has plants equal to the Highest Common Factor of 84, 126 and 189.

Finding the HCF: 84 = 7 × 12.

A glance at 126 tells us that it is not divisible by 12, so checking divisibility by 7, we have

126 = 7 × 18 and 189 = 7 × 27.

So obviously 7 is common to all three numbers and 12, 18 and 27 will yield another common factor i.e. 3.

Thus number of plants in each row = HCF of 84, 126 and 189 = 7 × 3 = 21.

Thus rows required will be $\frac{84}{21}+\frac{126}{21}+\frac{189}{21}$ i.e. 4 + 6 + 9 = 19 rows. We do not need to calculate this, we have already factorised 84, 126 and 189 and so use these factorised expressions to do the division.

LCM

The concept of LCM

Least Common Multiple of a set of numbers refers to the smallest number that can be completely divided by all the numbers in the given set.

Thus, if we are interested in finding the smallest number that can be completely divided by 6, 10 and 15, we are basically searching for the LCM of {6, 10, 15}.

Numbers that can be completely divided a given number *n*, are multiples of the number.

Numbers that are divisible by 6: 6, 12, 18, 24, 30, 36, 42, 48, 54, 60, 72, ……

These are ALL the numbers that can be divided by 6. There are no other numbers other than such multiples that can be completely divisible by 6.

Numbers that are divisible by 10: 10, 20, 30, 40, 50, 60, 70, ……

Numbers that are divisible by 15: 15, 30, 45, 60, 75, ……

The Common Multiples i.e. 30, 60, …… are the numbers that can be completely divided each of the three given numbers viz. 6, 10, 15.

Among these Common Multiples, the least is 30. Thus we can safely say that 30 is the least number that can be divided by each of 6, 10 and 15. There is no number less than 30 that can be divided by the given numbers. So the Least Common Multiple of 6, 10 and 15 is 30.

Thus if there are three clocks, each striking after 10 mins, 15 mins and 25 mins respectively and if they have just struck simultaneously, they will again strike simultaneously only after a common multiple of 10, 15 and 25 mins. Thus the time interval between two successive simultaneous strikes will be the Least Common Multiple of 10, 15 and 25 mins, i.e. 150 minutes. This example here forms the platform of applications of LCM. So be sure to understand it and the use of LCM completely before proceeding.

Finding the LCM:

Approach 1: Factorisation

Factorise each of the given number. The LCM will be the product of the highest powers of primes across the given numbers.

Find the LCM of 24, 40 and 54.

Factorising, $24 = 2^3 \times 3$

$40 = 2^3 \times 5$

$54 = 2 \times 3^3$

Since we are looking for a multiple of each of the following, we would have to take the highest power of each of the primes present in the above factorised forms.

Thus the LCM = $2^3 \times 3^3 \times 5$

Had we not take the highest power, we would not have got a common multiple. E.g. if we do not include 5 in the LCM, the number $2^3 \times 3^3$ would not have been a multiple of 40. Similarly had we just considered $2^2 \times 3^3 \times 5$, it would not have been a multiple of 24 (and nor of 40)

Approach 2: Taking the highest power of prime that appears in any one of the given numbers

Considering the above example of finding the LCM of 24, 40 and 54...

The highest power of 2 that appears in any of the numbers is 2^3 (appears in 24 and 40)

The highest power of 3 that appears in any of the numbers is 3^3 (appears in 54)

Also 5^1 appears in the number 40.

There are no further primes that are present in any of the numbers. Thus LCM = $2^3 \times 3^3 \times 5$

This approach is pretty useful if we want to find the LCM of many numbers and all the numbers are manageable numbers.

Find the LCM of 4, 5, 6, 7, 8, 9, 10, 12, 15, 16, 18, 20 and 25.

The highest power of 2 that appears in any of the number is 2^4.

The highest power of 3 that appears in any of the number is 3^2.

The highest power of 5 that appears in any of the number is 5^2.

The highest power of 7 that appears in any of the number is 7.

Since no other prime number is present in any of the numbers, the LCM is $2^4 \times 3^2 \times 5 \times 7$.

Approach 3: Multiple of the largest number

Start with the largest number and find which of its multiple is divisible by all.

Find the LCM of 15, 24, 40

The LCM has to be a multiple of 40 i.e. $40 \times n$. Find the least value of n that will make $40 \times n$ divisible by 15 and 24.

40×3 will be divisible by 15 and also by 24. Thus LCM is $40 \times 3 = 120$.

Applications of LCM:

1. Finding numbers that when divided by given divisors leave a constant remainder.

E.g. 9: Find the series of numbers that when divided by 12, 18 and 30 leave a remainder of 5 in each case.

These types of problems are characterised by the remainder being same in each case.

For such problems the series of numbers is the (LCM of divisors) × a + constant remainders, where a is any whole number 0, 1, 2, 3, ……

The LCM is the least number that is completely divisible by the divisors. If we add the required remainder to the LCM, this will always be left over as the remainder.

Thus, the series of numbers for this example are LCM (12, 18, 30) × a + 5 = 180a + 5.

The smallest such number is 5, second number is 185, third number is 365 and so on.

E.g. 10: n is a number that on being divided by 3, 4 and 6 leaves a remainder of 2 in each case.

i. Find the smallest three digit value that n can assume

ii. Find the largest three digit value that n can assume

LCM of 3, 4, and 6 is 12. Since the remainder is a constant, numbers of the required type are of the form 12×a + 2.

i. The smallest three digit number will be when a = 9 and the number is 110

ii. Multiple of 12 close to 999 is 996. Thus the required number is 998.

2. Finding numbers that when divided by given divisors leave remainders such that difference between divisor and remainder is constant.

E.g. 11: Find the series of numbers that when divided by 12, 18 and 30 leave a remainder of 5, 11 and 23 respectively.

These types of problems are characterized by the fact that the difference between the divisors and remainders is a constant. Thus in this case 12 – 5 = 18 – 11 = 30 – 23 = 7.

For such problems the series of numbers is the (LCM of divisors) × a – the constant difference, where a is any natural number 1, 2, 3, ……

Thus the series of numbers in this case is LCM(12, 18, 30) × a – 7 i.e. 180a – 7.

The smallest such number is 173, second smallest number is 353 and so on.

E.g. 12: n is a number that on being divided by 3, 4 and 6 leaves a remainder of 1, 2 and 4 respectively. Find the smallest four digit value that n can assume.

The LCM of 3, 4 and 6 is 12. Since the difference between the divisors and remainders s a constant 2 in each case, numbers satisfying the given condition are of the form 12×a – 2. To find a smallest four digit number of this type, we have to search for a multiple of 12 in the vicinity of 1000. A number divisible by 12 has to be divisible by 4 and 3. Thus the first such four digit number will be 1008. So the required answer is 1006.

Some General Observations:

1. Product of two numbers = HCF of the numbers × LCM of the numbers
2. HCF of a set of numbers has to be less than or equal to the least number in the set. Obviously, as it has to be a factor of the least number. Similarly LCM of a set of numbers has to be greater than or equal to the greatest number in the set.
3. HCF of a set of numbers has to be a factor of the LCM of the set of numbers.
4. HCF of fractions= $\frac{\text{HCF of numerators}}{\text{LCM of denominators}}$

   LCM of fractions= $\frac{\text{LCM of numerators}}{\text{HCF of denominators}}$

Note: The fractions have to be in the lowest reduced form.

Exercise

9. A milk man has to deliver 56 lts, 84 lts and 70 lts of milk to three different customers. What is the largest volume of measuring can that he can keep to measure the exact required quantity?

   a. 7 b. 8 c. 10 d. 14

10. n on dividing 50, 75 and 125 leaves a remainder of 2, 3 and 5 respectively. What is the largest value that n can take?

    a. 12 b. 36 c. 24 d. 8

11. A school has 120, 192 and 144students enrolled for its science, arts and commerce courses. All students have to be seated in rooms for an exam such that each room has students of only the same course and also all rooms have equal number of students. What is the least number of rooms needed?

    a. 13 b. 19 c. 11 d. 14

12. If the LCM of $2^3 \times 3^2$ and $2^a \times 3^b$ is $2^3 \times 3^3$, how many different values can each of a and b take?

    a. 4, 4 b. 3, 1 c. 4, 3 d. 4, 1

13. n is number that when divided by 5, 6 and 8 leave a remainder of 3 in each case. Find the smallest three digit value that n can take

    a. 133 b. 163 c. 123 d. 143

14. Find the largest three digit number that on being divided by 6, 10 and 15 leaves a remainder of 5 in each case

    a. 905 b. 995 c. 965 d. 985

15. When marbles from a bag were divided into groups of 8 marbles, three marbles were left; when groups of 10 were made, again three marbles were left; and when groups of 12 marbles were made, again three marbles were left out. What is the least number of marbles that the bag contained?

    a. 121 b. 243 c. 123 d. 63

16. n is a number that when divided by 3, 4 and 6 leaves a remainder of 1, 2 and 4 respectively. Find the smallest value that n can take.

a. 12 b. 10 c. 8 d. 14

17. What is the smallest number that on being divided by 9, 18, 24 leaves a remainder of 5, 14 and 20 respectively?

a. 72 b. 68 c. 76 d. 60

18. A number when divided by 4, 5, 6, 7, 8, 9 and 10 leaves a remainder of 3, 4, 5, 6, 7, 8 and 9 respectively. What is the smallest such number?

a. 2520 b. 2500 c. 2519 d. 2499

19. Find the HCF of $\frac{9}{10}, \frac{12}{25}, \frac{18}{35}, \frac{21}{40}$

a. 3/700 b. 3/1400 c. 9/1400 d. 9/700

20. Find the LCM of $\frac{1}{3}, \frac{5}{6}, \frac{2}{9}, \frac{4}{27}$

a. 60/3 b. 40/3 c. 20/3 d. 80/3

21. Find the GCD of 1.08, 0.36 and 0.9

a. 0.009 b. 0.09 c. 0.0009 d. 0.18

22. Find the LCM of 3, 2.7, 0.09

a. 27 b. 0.27 c. 9 d. 3.3

23. The HCF and LCM of two numbers is 11 and 7700 respectively. If one of the number is 275, find the other number.

a. 308 b. 154 c. 3388 d. 3300

Assignment: Number Systems

1. Two-third of six-eights of a number is 130. The number is:

(a) 65 (b) 130 (c) 260 (d) 300

2. $7\frac{2}{5}\times 2\frac{1}{2}+3\frac{2}{3}\div 2\frac{3}{4}=$

(a) $21\frac{1}{2}$ (b) $19\frac{1}{4}$ (c) $19\frac{5}{6}$ (d) 1

3. What should come in place of ? in the following equality?

$3\frac{1}{4}+5\frac{1}{2}+6\frac{3}{4}-4\frac{1}{2}=?+2\frac{1}{3}$

(a) $17\frac{1}{3}$ (b) $13\frac{2}{3}$ (c) 14 (d) $8\frac{2}{3}$

4. $\frac{\overline{1+1}\div 1-1\times\overline{1+1}+1-1+1}{\overline{2+2}\div 2-2+2\div 2+3}=$

(a) 1/2 (b) 1/3 (c) 2/3 (d) 1/4

5. If $\frac{3}{5}^{th}$ of a number is 23 more than 50% of the same number, then what will be $\frac{4}{5}^{th}$ of the number?

(a) 92 (b) 184 (c) 180 (d) 230

6. $\frac{2.333\times 2.333\times 2.333+1.667\times 1.667\times 1.667}{2.333\times 2.333-2.333\times 1.667+1.667\times 1.667}=$

(a) 4 (b) 7 (c) 1 (d) 6

7. Simplify: $\sqrt{5.5^2-4.4^2}\times\sqrt[2/3]{32\times 1125\div 36}$

(a) 3.3 (b) 33 (c) 23 (d) 2.5

8. $\sqrt[\frac{3}{2}]{\sqrt[3]{0.000064}}=$

(a) 0.08 (b) 0.008 (c) 0.02 (d) 0.002

9. $\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{3+\cfrac{1}{4+\cfrac{1}{5}}}}}=$

(a) 157/225 (b) 225/157 (c) 68/157 (d) 157/68

10. $\frac{1}{81^{-\frac{3}{4}}}-\frac{1}{125^{-\frac{2}{3}}}-\frac{1}{64^{-\frac{1}{6}}}=$

(a) −1 (b) 0 (c) 1/2 (d) 1

11. What should be the smallest number that should be added to 2476 to make it a perfect square?

(a) 29 (b) 124 (c) 24 (d) 75

12. If 24*a*15*b*, where *a* and *b* are digits, is divisible by 88, find the sum *a* + *b*.

(a) 7 (b) 8 (c) 9 (d) 10

13. A number N = 234*ab*, where *a* and *b* are the digits of the number. If the number N is divisible by 4 but not by 8, what is the sum of the digits of the largest value that N can assume?

(a) 20 (b) 21 (c) 22 (d) 24

14. A number N on being divided by 88 leaves a remainder of 63. When N is divided by 11, what will the remainder be?

(a) 3 (b) 8 (c) 9 (d) 14

15. A milk man has to deliver 56 lts, 84 lts and 70 lts of milk to three different customers. What is the largest volume of measuring can that he can keep to measure the exact required quantity?

(a) 4 (b) 7 (c) 8 (d) 14

16. Find the largest number that on dividing 85, 150 and 220 leaves a remainder of 1, 6 and 4 respectively.

(a) 5 (b) 10 (c) 12 (d) 15

17. What is the smallest number that needs to be added or subtracted to 123 so that the resulting number is perfectly divisible by 4, 5 and 6.

(a) 57 (b) 23 (c) 3 (d) 9

18. Radheshyam has some coins with him. When he tries to distribute the coins among 3 fakirs equally, he is left with 1 coin. On distributing the coins to 5 fakirs equally, he is left with 3 coins. Ans on distributing to 7 fakirs equally, he is left with 5 coins. Which of the following can be the number of coins that Radheshyam has?

(a) 1048 (b) 943 (c) 996 (d) 1024

19. LCM of two numbers is 576 and one of the numbers is 36. What can be the smallest other number?

(a) 64 (b) 112 (c) 96 (d) 108

20. The HCF and LCM of 60 and *x* is 20 and 240 respectively. Find the value of *x*.

(a) 10 (b) 20 (c) 40 (d) 80

Ratios

Ratios as Comparison

Ratios are used to compare two quantities. They give the relative 'size' of the two quantities. Thus when we say that the ratio of the amount that A and B has is 2 : 1, we are comparing the amounts that they have.

"A and B having amounts in ratio 2 : 1" is NOT same as "A and B having Rs. 2 and Re 1 respectively".

When we talk of ratios, we are essentially commenting on the '**relative**' amounts each has (and not the **actual** amounts) i.e. A has twice the amount that B has.

Hence, they having Rs. 2 and Re. 1 is just one possibility. They might also have Rs. 4 and Rs. 2. Or for that matter also Rs. 20 and Rs. 10.

In fact there can be any number of possible amounts that they have, with the only constraint that relatively, the amounts should conform to 2 : 1.

Mathematically speaking, the ratio $a : b$ is same as $\frac{a}{b}$. However this interpretation can be used only while considering ratios of two quantities.

Thus, the meaning of 'ratio of number of boys and girls in a class is 3 : 4' is $\frac{\text{number of boys}}{\text{number of girls}} = \frac{3}{4}$.

E.g. 1: What is the ratio of 1 hr 20 mins and 50 minutes?

When a ratio is to be found, both quantities must be in the same units. 1 hr 20 minutes is 80 mins. Thus the required ratio is $\frac{80 \text{ mins}}{50 \text{ mins}} = \frac{8}{5}$ i.e. 8 : 5.

E.g. 2: If the total number of students in a class is 40 and 25 of them are girls, find the ratio of the number of boys to the number of girls.

Since 25 students out of 40 are girls, the balance 40 – 25 = 15 will be boys.

Thus, the required ratio is 15 : 25. Cancelling out the common factor, 5, we get the ratio as 3 : 5.

Exercise

1. If $a : b$ is 3 : 4, find the ratio $(7a - 4b) : (3a + b)$.

   a. $\frac{16}{15}$ b. $\frac{5}{13}$ c. $-\frac{5}{13}$ d. $-\frac{16}{15}$

2. If $(7x - 4y) : (3x + y)$ is 5 : 13, find the ratio of x and y.

   a. $\frac{4}{3}$ b. $\frac{3}{4}$ c. $-\frac{4}{3}$ d. $-\frac{3}{4}$

3. If $\frac{2a^2 - 3b^2}{a^2 + b^2} = \frac{2}{41}$, find the ratio of $a : b$.

   a. 4 : 5 b. 5 : 4 c. 1 : 5 d. 1 : 4

4. If $x^2 + 4y^2 = 4xy$, then find the ratio $x : y$.

   a. 2 : 1 b. 1 : 2 c. 1 : 4 d. 4 : 1

Most often question would not be as easy as finding a ratio when the two quantities are given. More often, in the question a ratio is given and we need to find the underlying quantity.

Use of a constant or multiplying factor

Consider the data: ratio of boys and girls in a class is 3 : 4. From this can we find the number of boys and girls in the class?

As already learnt, the above data is just a comparison. The number of boys and girls could be 3 and 5; or they would be 30 and 50; or they could be 300 and 500.

The only condition is that if the number of boys is 3 × *k*, where *k* is any constant, then the number of girls NECESSARILY has to be 4 × *k*, where *k* is the same constant as earlier. Thus, while taking the ratio, $\frac{3 \times k}{4 \times k}$, the *k*'s will get cancelled and we will be left with $\frac{3}{4}$

If ratio of two or more quantities is given as *a* : *b* : *c* …, the quantities can be assumed as *ak*, *bk*, *ck*, ……

Question Type 1: A ratio with some additional information is given & one needs to find the underlying values.

E.g. 2: If the ratio of boys and girls in a class is 4 : 3 and the total number of students in the class is 84, find the number of boys and girls.

Assuming the number of boys and girls to be 4*k* and 3*k*, we have

$4k + 3k = 84$ ι.ε. $k = 12$

Thus the number of boys and girls is 48 and 36 respectively.

E.g. 3: If the ages of a husband, wife and their child are in the ratio 13 : 11 : 3 and the average age of the family of the three is 36 years, find the difference between the age of the husband and the wife?

The ages of the husband, wife and the child can be assumed as 13*k*, 11*k* and 3*k*.

Since the average age of the family is 30 years, $\frac{13k + 11k + 3k}{3} = 36 \Rightarrow 27k = 108$ i.e. $k = 4$.

Thus, the ages of the husband and wife are 13 × 4 and 11 × 4 i.e. 52 and 44 and the required difference is 8 years.

E.g. 4: The time taken by A and B to reach their office from home is in the ratio 7 : 10. If B takes 18 minutes more than A to reach office, find the time taken by each of them to reach office.

'18 minutes more' tells us that the difference in the time taken is 18 minutes.

Let time taken by A and B be 7*k* and 10*k* respectively. Thus, $10k - 7k = 18$ i.e. $3k = 18$ i.e. $k = 6$.

Thus, time taken by A and B are 7 × 6 = 42 mins and 10 × 6 = 42 mins.

Exercise

5. Find the sum of two numbers in the ratio 7 : 12 such that the greater exceeds the smaller by 275.

   a. 945 b. 1045 c. 1075 d. 1275

6. The ratio of a and b is 3 : 7. If a is 20 less than b, find the value of $a + b$.

   a. 20 b. 30 c. 40 d. 50

7. The ratio of the marks scored by Ram and Shyam is 3 : 7. If the average of Ram and Shyam's marks is 65, find the marks scored by them respectively.

   a. 31, 99 b. 91, 39 c. 39, 91 d. 29, 101

8. Three quantities are in the ratio of 2 : 3 : 4. If the difference between the first and third quantity is 8, find the value of the second quantity.

   a. 9 b. 3 c. 6 d. 12

9. Two friends separated by a certain distance start walking towards each other. When they meet one of them has walked 20 meters more than the other. If the ratio of the distances that each has covered is 2 : 3, find the distance that originally separated them.

   a. 100 b. 90 c. 120 d. 80

10. The price of a car was increased by Rs. 15000. Atul calculated that the ratio of the original price and new price is 26 : 27. Find the increased price of the car.

   a. 3,90,000 b. 4,05,000 c. 4,50,000 d. 3,75,000

Question Type 2: Dividing a given sum in a ratio

E.g. 5: Divide Rs. 72 in the ratio 2 : 3 : 4.

Method 1:

Let the three parts be $2k$, $3k$ and $4k$ respectively. The total of the three parts will be the total amount i.e. Rs. 72. Thus, $2k + 3k + 4k = 72$ i.e. $9k = 72$ i.e. $k = 8$.

Thus the three parts will be 2 × 8 = 16; 3 × 8 = 24; and 4 × 8 = 32.

Method 2:

If there were 2 + 3 + 4 = Rs. 9, each part would be Rs. 2, Rs. 3 and Rs. 4.

Thus, first part will be Rs. 2 out of every Rs. 9 i.e. $\frac{2}{9}^{th}$ of the total amount; second part will be Rs. 3 out of Rs. 9 i.e. $\frac{3}{9}^{th}$ of total amount; and third part will be Rs. 4 out of Rs. 9 i.e. $\frac{4}{9}^{th}$ of total amount.

Thus, required values will be $\frac{2}{9}^{th}$ of 72 = 16; $\frac{3}{9}^{th}$ of 72 = 24; and $\frac{4}{9}^{th}$ of 72 = 32.

When Ratios are in Fractions ...

Consider the following example which is exactly similar to the earlier one except that in this case, the ratio is given in terms of fractions.

E.g. 6: Divide Rs. 117 among A, B and C such that their shares are in the ratio $\frac{1}{2}:\frac{1}{3}:\frac{1}{4}$ respectively.

It is going to be a bit cumbersome to assume the individual shares as $\frac{1}{2}k, \frac{1}{3}k, \frac{1}{4}k$. An alternative is to manipulate the given ratio such that we deal in natural numbers.

Multiplying all numbers of a ratio by same number leaves the ratio unchanged

It is very obvious that each of 30 : 40 or 300 : 400 or 60 : 80 is same as 3 : 4. Thus, multiplying each number in a ratio by the same number leaves the ratio unchanged. When we multiply each number in a ratio, in effect we are just increasing the magnitude of all the numbers and since it is the same number, relatively, the numbers are still same in comparison.

We can multiply all three with a same number such that the fractions turn out to be natural numbers. Needless to explain, the number has to be a multiple of 2, 3 and 4 and 12 would be a right choice in this case. The given ratio is same as $\frac{1}{2}\times 12:\frac{1}{3}\times 12:\frac{1}{4}\times 12$ i.e. 6 : 4 : 3.

Now, 117 is the sum of the three shares i.e. the sum in ratio scale, 13, corresponds to 117. So k = 9.

The individual shares are 6 × 9, 4 × 9, 3 × 9 i.e. 54, 36 and 27.

When ratios are given to be equal

E.g. 7: Divide Rs. 117 among A, B and C such that $\frac{\text{A's share}}{2}=\frac{\text{B's share}}{3}=\frac{\text{C's share}}{4}$.

Assuming $\frac{\text{A's share}}{2}=\frac{\text{B's share}}{3}=\frac{\text{C's share}}{4}=k$, we have A's share = 2k, B's share = 3k and C's share = 4k. Thus, the share of A, B and C are in the ratio 2 : 3 : 4. (this ratio is different from that obtained in earlier example namely 6 : 4 : 3)

The sum of the share, in the ratio scale, 9, corresponds to 117 i.e. k = 13. So the individual shares are 2 × 13, 3 × 13, 4 × 13 i.e. 26, 39 and 52.

Don't confuse ...

... "A, B and C's shares are in the ratio $\frac{1}{2}:\frac{1}{3}:\frac{1}{4}$" and the data that "$\frac{\text{A's share}}{2}=\frac{\text{B's share}}{3}=\frac{\text{C's share}}{4}$".

In the former case, since ratios are given in fractions, $\frac{1}{2}:\frac{1}{3}:\frac{1}{4}$, we multiply with the LCM of denominators to get rid of the fractions. Thus, A, B and C's share are in ratio 6 : 4 : 3.

In the latter case quite a few students see the 2, 3 and 4 in the denominator and hence again think of multiplying with the LCM, which is a misleading path. One just needs to equate $\frac{\text{A's share}}{2}=\frac{\text{B's share}}{3}=\frac{\text{C's share}}{4}$ to k and one gets A, B and C's share are in ratio 2 : 3 : 4.

In fact the data "2 × A'share = 3 × B's share = 4 × C's share" will result in A, B and C's share being in ratio $\frac{1}{2}:\frac{1}{3}:\frac{1}{4}$ i.e. 6 : 4 : 3.

Thus, $\frac{\text{A's share}}{x} = \frac{\text{B's share}}{y} = \frac{\text{C's share}}{z}$ means the shares are in ratio $x : y : z$

And x × A's share = y × B's share = z × C's share means the shares are in ratio $\frac{1}{x}:\frac{1}{y}:\frac{1}{z}$

Exercise

11. Divide 2220 in the ratio $\frac{1}{4}:\frac{1}{5}:\frac{1}{6}$.

    a. 600, 800, 720 b. 900, 720, 600 c. 900, 700, 620 d. 720, 600, 800

12. If $3A = 4B = 5C = 6D$ and $A + B + C + D = 1026$, find the value of A.

    a. 360 b. 270 c. 216 d. 180

13. All the 585 students of a school are divided into three different groups such that the half the number of students in first group, one-third the number of students in second group and one-fourth the number of students in third group are equal. Find the number of students in each group.

    a. 130, 195, 260 b. 120, 205, 260 c. 150, 175, 260 d. 130, 205, 250

14. A amount of Rs. 735 was divided between A, B and C. If each of them had received Rs. 25 less, their shares would be in the ratio of 1 : 3 : 2. Find the share of C.

    a. 220 b. 245 c. 270 d. 355

15. Rs. 5390 is divided into three parts in such a way that half of the first part, one-third of the second part and one-sixth of the third part are in the ratio 2 : 3 : 6. Find the third part.

    a. 440 b. 990 c. 3960 d. 1430

16. Divide Rs. 1331 into three parts in such a way that half of the first part, one-third of the second part and one-sixth of the third part are in the ratio 6 : 3 : 2.

    a. 484, 363, 484 b. 363, 484, 484 c. 484, 484, 363 d. 163, 584, 584

17. Rs. 3280 is divided among A, B, C and D such that A gets twice of what B gets, B gets thrice of what C gets and C gets four times the amount that D gets. Find C's share.

    a. 1920 b. 960 c. 320 d. 80

18. Rs. 352 is divided among A, B, C and D such that A gets as much as B and C get, B gets as much as C and D get and C gets twice of what D gets. Find B's share.

    a.160 b. 96 c. 64 d. 32

Question Type 3: Given a : b and b : c, finding a : c

In such cases, the first ratio gives us the relation between *a* and *b*; the second ratio gives us the relation between *b* and *c*; thus logic says that we should be able to find the relation between *a* and *c* because <u>*b* is common to the two relation</u>. And *b* being common to the two relations is exactly what we use to find the answer, as done in the following examples.

E.g. 8: If *a* : *b* is 3 : 4 and *b* : *c* is 5 : 6, find the ratio *a* : *b* : *c*.

Since *b* is common to the two ratios, we should make the numeric value of *b* the same in both the ratios. Also remember that when all terms of a ratio are multiplied with the same constant, the ratio does not change.

Thus in the first ratio *b* could be changed to any multiple of 4 and in the second ratio *b* could be changed to any multiple of 5. Thus, we should make *b* a multiple of 4 and 5 i.e. 20.

a : *b* is 3 × 5 : 4 × 5 i.e. 15 : 20 *b* : *c* is 5 × 4 : 6 × 4 i.e. 20 : 24

Thus when *b* is 20, *a* is 15 and *c* is 24 and the required ratio of *a* : *b* : *c* is 15 : 20 : 24.

E.g. 9: If *a* : *b* is 2 : 5, *b* : *c* is 3 : 5 and *c* : *d* is 5 : 4, find the ratio *a* : *b* : *c* : *d*.

In the ratios of *a* : *b* and *b* : *c*, if *b* has to be the same numeric value, it should be a multiple of 5 and 3 i.e. 15. Thus,

a : *b* is 2 : 5 which is same as 6 : 15 *b* : *c* is 3 : 5 which is same as 15 : 25.

Thus *a* : *b* : *c* is 6 : 15 : 25

Now *c* is common to *a* : *b* : *c* and *c* : *d* and thus making *c* equal in the two ratios,

c : *d* is 5 : 4 which is same as 25 : 20. Thus *a* : *b* : *c* : *d* is 6 : 15 : 25 : 20

In the above question had the question been finding just *a* : *d*, one could have avoided all this work and just done the following: $\frac{a}{d} = \frac{a}{b} \times \frac{b}{c} \times \frac{c}{d} = \frac{2}{5} \times \frac{3}{5} \times \frac{5}{4} = \frac{3}{10}$

Please be careful of which term to be made numerically equal. It should be that term which appears in both the relations.

E.g. 10: Given *a* : *b* is 6 : 5 and *a* : *c* is 15 : 11, find the ratio *a* : *b* : *c*.

In this example, we would make *a* equal in the two ratios as it is common to both the relations. Now, *a* could be any multiple of 6 in first ratio and any multiple of 15 in second ratio, thus let's make *a* equal to 30

a : *b* is 6 : 5 i.e. 30 : 25 *a* : *c* is 15 : 11 i.e. 30 : 22

Thus *a* : *b* : *c* is 30 : 25 : 22

Exercise

19. If *a* : *b* is 2 : 3, *b* : *c* is 4 : 5 and *c* : *d* is 6 : 7, find the ratio of *a* : *b* : *c* : *d*

a. 2 : 3 : 4 : 5 b. 4 : 5 : 6 : 7 c. 2 : 3 : 6 : 7 d. 16 : 24 : 30 : 35

20. If *a* : *b* is 3 : 4; *b* : *c* is 5 : 3 and *a* : *d* is 2 : 5, find the ratio of *c* : *d*.

a. 3 : 5 b. 8 : 15 c. 1 : 5 d. 5 : 8

21. If $A : B = \frac{1}{2} : \frac{3}{8}$, $B : C = \frac{1}{3} : \frac{5}{9}$ and $C : D = \frac{5}{6} : \frac{3}{4}$, then find the ratio of *A* : *B* : *C* : *D*.

a. 4 : 3 : 5 : 10 b. 4 : 3 : 5 : 9 c. 8 : 6 : 10 : 9 d. 8 : 6 : 10 : 5

Few Standard Questions on Ratios:

Problems on Ages

E.g. 11: The ratio of the present ages of a father and a son is 8 : 1. Eight years hence, the ratio would be 10 : 3. What is the present age of the father?

Assuming the present ages of the father and son to be $8k$ and k respectively, the ages after 8 years will be $8k + 8$ and $k + 8$. Thus we have,

$\frac{8k+8}{k+8} = \frac{10}{3}$ i.e. $24k + 24 = 10k + 80$ i.e. $14k = 56$ i.e. $k = 4$.

Thus present age of father is $8 \times 4 = 32$ years.

Problems on Incomes and Expenditures:

E.g. 12: If the incomes of A and B are in ratio 6 : 5 and their expenses are in ratio 5 : 4, find the ratio of their savings.

Error-prone area

When two ratios are given, the constants need to be taken as different

Quite a few students assume the income as 600 and 500 and the expenses as 500 and 400 and find the savings as 100 in each case and thus, conclude the answer to be 1 : 1.

Few others, approach it mathematically and yet arrive at the same erroneous answer. The incomes are assumes as $6k$ and $5k$, expenses are assumed as $5k$ and $4k$ and the ratio of savings is found as $(6k - 5k)$: $(5k - 4k)$ i.e. $k : k$ i.e. 1 : 1.

The above answer is wrong as evident from the data that the incomes are 600 and 500 (in the ratio 6 : 5) and the expenses are 50 and 40 (in the ratio 5 : 4). In this case, their savings are 550 and 460 which are not in the ratio 1 : 1.

There are two ratios, ratio of income 6 : 5 and ratio of expenses 5 : 4. And the two ratios are independent of each other. Thus, if the income is assumed as $6k$ and $5k$, i.e. some multiple of 6 : 5, the expenses need not be the same multiple of 5 : 4. Thus, the multiple of 5 : 4, that needs to be used has to be different from the earlier multiple, say $5n$ and $4n$.

Assuming the incomes as $6k$ and $5k$ and the expenses as $5n$ and $4n$, we get the ratio of savings as $\frac{6k-5n}{5k-4n}$, the value of which will keep changing based on the relative values of k and n. Thus, there is no unique answer to the ratio of savings, as must be evident from the two specific cases discussed in the box above – with incomes as 600 & 500 and expenses as 500 & 400, the ratio of savings is 1 : 1; and with incomes as 600 & 500 and expenses as 50 & 40, the ratio of savings is 55 : 46.

What if both ratio of income and savings is 6 : 5 and 6 : 5.

Approaching the problem mathematically, assuming incomes as $6k$ and $5k$ and expenses as $6n$ and $5n$, the ratio of savings is $\frac{6k-6n}{5k-5n} = \frac{6(k-n)}{5(k-n)} = \frac{6}{5}$. (The above is true when each of them save something i.e. except the case when income = expenditure i.e. when except when $k = n$. In this case, the savings of each will be 0)

Thus, we see that in case both income and expenses are in ratio $a : b$, then savings will also be in the ratio $a : b$. And in case the ratio of income and expenses is different, the ratio of savings cannot be determined.

E.g. 13: The ratio of the income and expenditure of Amit and Sumit are 5 : 8 and 11 : 10. If Amit and Sumit save Rs. 7900 and Rs. 21000 respectively, find their incomes.

We have already learnt that two different constant of proportionality for the two ratios will have to be used.

Thus assuming the income of Amit and Sumit as $5k$ and $8k$ and assuming the expenditure of the two as $11n$ and $10n$, we have the following two equations:

$5k - 11n = 7900$ …… (i)

$8k - 10n = 21000$ …… (ii)

Solving the two equations simultaneously and eliminating n we get,

(i) × 10 $50k - 110n = 79000$ ……(iii)

(ii) × 11 $88k - 110n = 231000$ …… (iv)

(iv) – (iii) $38k = 152000$ i.e. $k = 4000$.

Thus the incomes of Amit and Sumit are 5 × 4000 = 20,000 and 8 × 4000 = 32,000 respectively.

E.g. 14: A bag has Re. 1, 50 ps and 25 ps coins. If the ratio of the number of coins of the respective denominations is 3 : 2 : 8 and the bag has a total amount of Rs. 156, find the number of 50 paise coins.

Since the ratio is given of the number of coins, let the number of coins be $3k$, $2k$ and $8k$. Remember these are the number of coins and equating $3k + 2k + 8k = 156$ WILL BE WRONG. The left hand side of this equation is the total number of coins and the right hand side, 156, is NOT the total number of coins but is the value of the number of coins.

Thus, the number of coins will have to be transferred to value of coins, in Rs.

$3k$ Re.-1 coins will amount to $3k \times 1$ = Rs. $3k$

$2k$ 50-paise coins will amount to $2k \times 0.5$ = Rs. k

$8k$ 25-paise coins will amount to $8k \times 0.25$ = Rs. $2k$.

Thus, total amount in bag will be $3k + k + 2k$ i.e. $6k$ and this will be equal to Rs. 156. Thus, $6k = 156$ i.e. $k = 26$.

We want to find the number of 50-paise coins and the required answer will be $2k$ i.e. 2 × 26 = 52 coins.

Exercise

22. The salary of Rahul and Priyanka are in the ratio of 6 : 5 whereas their expenses are in the ratio 4 : 3. Find the ratio of their saving?

    a. 1 : 1   b. 5 : 4   c. 2 : 1   d. Cannot be determined

23. The salary of Rahul and Priyanka are in the ratio of 6 : 5 and their expenses are also in the ratio 6 : 5. Find the ratio of their saving?

    a. 1 : 1   b. 11 : 1   c. 1 : 11   d. 6 : 5

24. Two numbers are in the ratio 3 : 4. If 7 is subtracted from each of the numbers, the remainders are in the ratio 2 : 3. Find the sum of the numbers.

    a. 49   b. 35   c. 25   d. 42

25. Present ages of Sameer and Anand are in the ratio 5 : 4. Three years hence, the ratio of their ages will become 11 : 9. What is Anand's present age?

a. 16 b. 20 c. 24 d. 28

26. The incomes of A and B are in the ratio 3 : 2 and their expenses are in the ratio 5 : 3. If each saves Rs. 1000, then find the income of A.

a. 5000 b. 6000 c. 4000 d. 3000

27. The total income of A, B and C is Rs. 80,000. Their expenditure are Rs. 8,000, Rs. 12,000 and Rs. 15,000 respectively. If their savings are in the ratio of 2 : 3 : 4, find their incomes (in thousands).

a. 10, 15, 20 b. 18, 27, 35 c. 15, 10, 20 d. 20, 10, 15

28. The number of 1 Re, 50 paise and 25 paise coins in a bag is in the ratio 3 : 4 : 5. If the bag has Rs. 300, find the number of 50 paise coins.

a. 60 b. 96 c. 144 d. 192

29. A sum of Rs. 1300 is divided among P, Q, R and S such that $\frac{\text{P's share}}{\text{Q's share}} = \frac{\text{Q's share}}{\text{R's share}} = \frac{\text{R's share}}{\text{S's share}} = \frac{2}{3}$.

Find P's share.

a. 240 b. 160 c. 80 d. 120

30. If $(a + b) : (b + c) : (c + a)$ is 3 : 4 : 5 and if $a + b + c = 18$, find the value of $a \times b \times c$.

a. 60 b. 120 c. 144 d. 162

Partnership

Partnership is when two or more people pool in money as capital for a common venture. The profit of the venture is then divided among the people depending on the amount of money that each has invested. Quite often the time duration for which an individual has invested money in the venture may also be different for different partners – one may quit the partnership withdrawing his investment prematurely or a new partner may join the venture sometimes later. If a partner quits, the venture does not cease to exist. After a pre-defined interval of time, the profits are distributed and the partner who exited early (or entered later) would consequently get a diminished share of profit as compared to that had he been there for complete duration.

The following lists out how the profit is divided in each of the different scenarios possible.

Different Investments, Same Time period of Investing

If the time period for which the partners have invested the money is same, the profits is distributed among the partners in the ratio of their investments

If the amount invested by the partners are C_1, C_2, C_3 then the profit is distributed in the ratio $C_1 : C_2 : C_3$.

E.g. 1: Rahul and Rohit get in Rs. 4000 and Rs. 5000 to fund a new venture. In what ratio should they divide the profit of Rs. 1,80,000 earned at the end of the year?

The profit has to be divided among Rahul and Rohit in the ratio of their investments i.e. 4 : 5. Thus, let Rahul's and Rohit's share of profit be $4k$ and $5k$ respectively. The two share together would be the entire profit i.e. $4k + 5k = 1{,}80{,}000$ i.e. $9k = 1{,}80{,}000$ i.e. $k = 20{,}000$.

Thus Rahul's share = 4 × 20,000 = 80,000 and Rohit's share = 5 × 20,000 = 1,00,000.

Same Investments, different Time periods

If the investments made by the partners are same, but the time period is different, the profit is divided in the ratio of the time periods.

When the investment and also the time period is different

In cases of partnership when both the investment and the time period the amount is invested are different for the partners, the profit is shared in the ratio of their investments × time.

Let there be three partners, one invests C_1 for t_1 time, second invests C_2 for t_2 time and third invests C_3 for t_3 time. The profit is shared in the ratio $C_1 \times t_1 : C_2 \times t_2 : C_3 \times t_3$.

E.g. 2: A and B enter into a partnership with Rs. 8,000 and 15,000 respectively. After 3 months C joins them by investing Rs. 10,000. 4 months before the first year is completed, B quits, taking his invested amount back with him. In what ratio should the profit of Rs. 2,55,000 earned in the first year be distributed among the three?

A has invested 8,000 for 12 months

B has invested 15,000 for 8 months

C has invested 10,000 for 9 months.

Thus the profit has to be distributed in the ratio of 8 × 12 : 15 × 8 : 10 × 9 i.e. 16 : 20 : 15

A's share = $\frac{16}{51} \times 2{,}55{,}000 = 16 \times 5{,}000 = 80{,}000$

B's share = $\frac{20}{51} \times 2{,}55{,}000 = 1{,}00{,}000$ C's share = $\frac{15}{51} \times 2{,}55{,}000 = 75{,}000$

When one partner has different amounts invested in different time periods.

Let's say in a partnership between A and B, A invests Rs. I_a for a time period of t_a. But B invests Rs. I_{b1} for a period of t_{b1} time and Rs. I_{b2} for a period of t_{b2} time. In this case the profit will be divided between A and B in the ratio $I_a \times t_a : (I_{b1} \times t_{b1} + I_{b2} \times t_{b2})$

E.g. 3: A, B and C enter into a partnership. They invest Rs. 40,000, Rs. 80,000 and Rs. 1,20,000 respectively. At the end of the first year, B withdraws Rs. 40,000 while at the end of second year, C withdraws Rs. 80,000. In what ratio will the profit be shared at the end of three years?

The profit has to be shared in the ratio of (40 × 3) : (80 × 1) + (40 × 2) : (120 × 2) + (40 × 1)

i.e. 120 : 160 : 280 i.e. 3 : 4 : 7.

Different ways of accounting for B's investment

One could have thought of B's investment in different ways, though all of them yield the same end-result ….

B invested 80,000 for 1 year and then 40,000 for next two years: Treated mathematically as 80 × 1 + 40 × 2 i.e. 160.

B invested 40,000 for the entire 3 year period and an additional 40,000 for the first year: Treated mathematically as 40 × 3 + 40 × 1 i.e. 160.

B invested 80,000 for entire 3 years less 40,000 for last two years: Treated mathematically as 80 × 3 – 40 × 2 i.e. 160.

Working Partner drawing a salary

Sometimes, one partner may be an active manager and hence in addition to taking a share of profit, he would also be entitled to a salary. In such cases, the salary of the partner is first deducted from the profits and the residual profit amount is then distributed in a manner as stated in the earlier methods.

E.g. 4: Two partners, A and B invest Rs. 10,000 and Rs. 15,000 in a partnership firm which makes a profit of Rs. 50,000 at the end of the year. But since A is a working partner, he is entitled to a salary of Rs. 20,000 for the year. What is the amount that B receives?

The residual profit after A's salary = Rs. 50,000 – Rs. 20,000 = Rs. 30,000. This amount will now be divided among A and B in the ratio of their investments i.e. in the ratio 2 : 3. Thus, amount received by B = $\frac{3}{5}^{th}$ of Rs. 30,000 i.e. Rs. 18,000.

Exercise

1. A starts a business with Rs. 3,500 and after 5 months, B joins A. At the end of the year, the profit is divided between A and B in the ratio 2 : 3. What amount did B invest in the business?

   a. 2800 b. 2700 c. 1800 d. 9000

2. Three partners shared the profit in a business in the ratio 5 : 7 : 8. They had partnered for 14 months, 8 months and 7 months respectively. What was the ratio of their investments.

   a. 5 : 4 : 4 b. 20 : 49 : 64 c. 784 : 320 : 245 d. 19 : 15 : 15

3. A, B and C rent a pasture. A puts 10 oxen for 7 months, B puts 12 oxen for 5 months and C puts 15 oxen for 3 months for grazing. If the rent of the pasture is Rs. 175, find the share of C in the rent.

   a. 45 b. 105 c. 30 d. 70

4. A and B entered into a partnership investing Rs. 16,000 and Rs. 12,000 respectively. After 3 months, A withdrew Rs. 5,000 while B invested Rs. 5,000 more. After 3 more months, C joins the business with a capital of Rs. 21,000. If the total profit at the end of the year is Rs. 26,400, by what amount does B's share exceed the share of C.

   a. 3025 b. 2695 c. 5500 d. 3465

5. A and B enter into a partnership by contributing Rs. 1,00,000 and 1,50,000 respectively. After 3 months C joins them by contributing a capital of 2,00,000. 4 months before the end of the year B quits taking his share of the capital away with him. Because of this A adds another 50,000 of capital from his side. At the end of the year the partnership makes a profit of Rs. 1,65,000. What will be the difference between C's share and A's share of the profit?

   a. 11,000 b. 13,000 c. 15,000 d. 8,250

Proportion

When two ratio are equal, $\frac{a}{b}=\frac{c}{d}$, then *a, b, c, d* are said to be in proportion.

Conversely, if *a, b, c, d* are in proportion, then we have $\frac{a}{b}=\frac{c}{d}$.

a & *d* are called the extremes and *b* & *c* are called the means.

From the above relation it should be obvious that if *a, b, c* and *d* are in proportion, then $a \times d = b \times c$ i.e. product of extremes = product of means.

The order in a proportion is important. Thus, *a, b, c* and *d* are in proportion and *a, b, d, c* are in proportion will give different results.

Continued proportion

If *a, b, c, d, e,* …… are in continued proportion, then we have $\frac{a}{b}=\frac{b}{c}=\frac{c}{d}=\frac{d}{e}=\ldots\ldots$

In specific, if three numbers, *a, b, c* are in continued proportion, then $b^2 = a \times c$. In this case *b* is called the mean proportional to *a* & *c*.

Operations on Proportions

If we have $\frac{a}{b}=\frac{c}{d}$, then each of the following also is true:

$\frac{b}{a}=\frac{d}{c}$ (Invertendo)

$\frac{a}{c}=\frac{b}{d}$ (Alternendo)

$\frac{a+b}{b}=\frac{c+d}{d}$ (Componendo, this is obtained by adding 1 to both sides of original equality)

$\frac{a-b}{b}=\frac{c-d}{d}$ (Dividendo, this is obtained by subtracting 1 to both sides of original equality)

$\frac{a+b}{a-b}=\frac{c+d}{c-d}$ (Componendo and Dividendo, obtained by dividing the above two equalities)

Logic

The relation found by componendo can be roved by adding 1 to both sides of the proportion …

If $\frac{a}{b}=\frac{c}{d}$ is given. Adding 1 to both sides, $\frac{a}{b}+1=\frac{c}{d}+1$ i.e. $\frac{a+b}{b}=\frac{c+d}{d}$.

Similarly the relation found by dividendo can be proved by subtracting 1 from both sides of the proportion.

And taking the ratio of the results of componendo and that of dividendo will result in the relation found by componendo & dividendo.

The above have very limited usage and those limited scenarios can also be solved further without the use of the above.

Application

When fractions of the type $\frac{x+y}{x-y}$ are involved, performing componendo & dividendo on this fraction directly yields $\frac{x}{y}$. And we will find such fractions in couple of scenarios (boats & streams, relative speed in case of opposite & same direction, etc)

Thus, if $\frac{3x-4y}{3x+4y}=\frac{7}{2}$, rather than cross-multiplying & re-arranging, once just perform componendo & dividendo to result in $\frac{3x}{4y}=\frac{9}{5}$

All of these have very limited applications. Componendo and Dividendo can be used in the following situation:

E.g. 1: If $\frac{4a+5b}{4a-5b}=\frac{7}{5}$, find $a : b$

Performing componendo and dividendo, we have

$$\frac{(4a+5b)+(4a-5b)}{(4a+5b)-(4a-5b)}=\frac{7+5}{7-5}$$

$$\therefore \frac{8a}{10b}=\frac{12}{2}$$

$$\therefore \frac{a}{b}=\frac{15}{2}$$

One could also have found this out by cross multiplying as follows:

$$\frac{4a+5b}{4a-5b}=\frac{7}{5} \Rightarrow 20a+25b=28a-35b$$

$$\therefore 60b=8a \text{ and } \frac{a}{b}=\frac{60}{8}=\frac{15}{2}$$

E.g. 2: If the sum of two numbers and the difference of the numbers are in the ratio 8 : 5, find the ratio of the numbers.

Noticing the sum and the difference, one should have thought of $(x + y)$ and $(x - y)$ immediately. And to find the ratio $x : y$ from ratio of above, one just needs to do componendo & dividendo. Thus the required answer is 13 : 3.

Law of Equal Ratios

If $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=......=k$, then by law of equal ratios, $\frac{a+c+e}{b+d+f}=k$

In fact not just this ratio, any identical 'linear' combination of numerators and denominators will also be k or in other words will be equal to each of $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}$ i.e. each of $\frac{a-c+e}{b-d+f}, \frac{a-c-e}{b-d-f}, \frac{2a-3c}{2b-3d}, \frac{5c-a+2e}{5d-b+2f}$ or any such relation will also be equal to k or will be equal to each of $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}$.

Logic

Let $\frac{a}{b} = \frac{c}{d} = \frac{e}{f} = ... = k$. Thus, $a = bk$, $c = dk$, $e = fk$.

Consider $\frac{5c - a + 2e}{5d - b + 2f}$. It can be written as $\frac{5dk - bk + 2fk}{5d - b + 2f}$ i.e. $\frac{k(5d - b + 2f)}{5d - b + 2f}$ i.e. k.

In fact, while the above limits the application to any 'linear' combination i.e. no variable can be multiplied with each other nor can their power be taken

E.g. 3: If $\frac{a}{b} = \frac{c}{d} = \frac{4}{5}$, find the ratio $\frac{a+c}{b+d}$.

By using the property explained earlier, $\frac{a}{b} = \frac{c}{d} = \frac{a+c}{b+d} = \frac{4}{5}$

The above property can also be expanded to a more generic case as follows:

If $\frac{a}{b} = \frac{c}{d} = \frac{e}{f} = ...$, then

the ratio $\frac{k_1 \times a + k_2 \times c + k_3 \times e......}{k_1 \times b + k_2 \times d + k_3 \times f......}$ is also equal to each of the above ratio, where k_1, k_2 and k_3 could be any real numbers.

E.g. 4: If $\frac{a}{b} = \frac{c}{d} = \frac{e}{f} = \frac{4}{5}$, find the ratio $\frac{2a + 3c - e}{2b + 3d - f}$.

By using the property explained earlier, $\frac{a}{b} = \frac{c}{d} = \frac{e}{f} = \frac{2a + 3c - e}{2b + 3d - f} = \frac{4}{5}$

E.g. 5: If $\frac{a}{b} = \frac{c}{d} = \frac{4}{5}$, find the ratio $\frac{a^2 - c^2}{b^2 - d^2}$. (Assume $a \neq c$)

$\frac{a}{b} = \frac{c}{d} = \frac{4}{5}$ i.e. $\frac{a^2}{b^2} = \frac{c^2}{d^2} = \frac{4^2}{5^2} \Rightarrow \frac{a^2 - c^2}{b^2 - d^2} = \frac{16}{25}$

For those who need to work it out, plugging $a = \frac{4}{5}b$ and $c = \frac{4}{5}d$, we have

$$\frac{a^2 - c^2}{b^2 - d^2} = \frac{\left(\frac{4}{5}b\right)^2 - \left(\frac{4}{5}d\right)^2}{b^2 - d^2} = \frac{\left(\frac{4}{5}\right)^2 (b^2 - d^2)}{(b^2 - d^2)} = \left(\frac{4}{5}\right)^2$$

E.g. 6: If $\frac{a}{b} = \frac{c}{d} = \frac{4}{5}$, find the ratio $\frac{4a - 3c}{5b + 4d}$

In this example the relation asked $\frac{4a - 3c}{5b + 4d}$ cannot be arrived at by using the property we learnt.

Assuming values for a, b, c, and d would be disastrous

If $a = c = 4$ and $b = d = 5$ and then the ratio

$\frac{4a-3c}{5b+4d} = \frac{4\times4-3\times4}{5\times5+4\times5} = \frac{16-12}{25+20} = \frac{4}{45}$ would be WRONG, because with a different set of values of a, b, c and d we would get a different answer, as follows:

The values $a = 4$, $b = 5$ and $c = 8$ and $d = 10$ also satisfy the given condition. But in this case,

$$\frac{4a-3c}{5b+4d} = \frac{4\times4-3\times8}{5\times5+4\times10} = \frac{16-24}{25+40} = \frac{-8}{65}$$

Without assuming values, $a = \frac{4}{5}\times b$ and $c = \frac{4}{5}\times d$. Substituting these, we get

$\frac{4a-3c}{5b+4d} = \frac{4\times\left(\frac{4}{5}\times b\right)-3\times\left(\frac{4}{5}\times d\right)}{5b+4d} = \frac{16b-12d}{25b+20d}$, the value of which will depend on the values of b and d. Thus the answer to this question is that there is no unique value for the required ratio or cannot be determined.

Exercise

1. What number must be subtracted from each of 7, 8, 11 and 14, so that the remainders are proportional.

   a. 1 b. 2 c. 3 d. 5

2. What number must be added to each of 13, 25 and 45 such that the results are in continued proportion.

   a. 1 b. 2 c. 3 d. 5

3. Find the fourth proportional to 3, 5, 27.

   a. 45 b. 81/5 c. 5/9 d. 9/5

4. Find the mean proportional between 6 and 24.

   a. 9 b. 12 c. 15 d. 18

5. If $\frac{1}{5} : \frac{1}{x} = \frac{1}{x} : \frac{1}{125}$, then find the value of x.

   a. 10 b. 15 c. 25 d. 5

6. If $\frac{a}{x-y} = \frac{b}{y-z} = \frac{c}{z-x}$, where none of the individual ratio is 0, find the value of $a + b + c$

   a. –1 b. 0 c. 1 4. $x + y + z$

Variation - Chain Rule

If 10 hens lay 10 eggs in 10 days, how many eggs will 1 hen lay in 1 day?

The above type of questions where a set of data is given (10 hens, 10 days, 10 eggs) and question is asked based on another instance of the same data but with one unknown variable (1 hen, 1 day, ? egg) are commonly called as 'Chain Rule' (when we solve the question, the reason why chain rule will be apparent.

While these questions are based on direct and inverse proportionality, they can be solved very elegantly without being aware of the theory and approaching the problems logically. They are an easier lot and also very easily identifiable as being questions on chain rule. So let us learn the logical way to solve the questions and later look at the theory. Rest assured you will not need any theory.

Let's start with a simple question.

If 10 weavers weave 10 carpets in 10 days, how many carpets will 1 weaver weave in 1 day?

If you have got your answer as 1 carpet, make sure you read the following very carefully, as it is not the correct answer.

10 weaver, 10 carpets, 10 days is one instance of the variables involved.

Had there been just 1 weaver i.e. the number of weavers reduce to $\frac{1}{10}$, obviously the number of carpets weaved would also reduce (proportionally) i.e. carpets weaved would become $10 \times \frac{1}{10}$ i.e. 1 carpet.

Thus... 1 weaver weaves 1 carpet in 10 days.

If the number of days is reduced, from earlier 10 days to just 1 day now, obviously the 1 weaver would have weaved lesser carpet, to be exact $\frac{1}{10}$ of the carpet.

Thus, 1 weaver would weave $\frac{1}{10}^{th}$ carpet in 1 day.

Let's take another example:

E.g. 1: 4 carpenters make 20 chairs in 5 days. How many chairs will 8 carpenters make in 10 days?

4 carpenters make 20 chairs in 5 days. So 8 carpenters will make double the number of chairs in 5 days (same time as earlier) i.e. $20 \times \frac{8}{4} = 40$ chairs.

When you are accounting for the change in the number of chairs made because the number of carpenters has increased, you have to keep all the other factors (days available, etc) constant. Their effect will be included when that particular variable is taken as the focus.

Next since the number of days has also increased from 5 to 10 i.e. doubled, these 8 carpenters who earlier made 40 chairs in 5 days will make double the number of chairs in 10 days i.e. $40 \times \frac{10}{5} = 80$ chairs.

E.g. 2: 6 labourers can build a wall in 10 days. How many labourers of double the efficiency, are needed to build 3 such walls in 5 days?

To build 1 wall in 10 days, 6 laborers are required, so to build 3 walls in 10 days, laborers will be required.

If the number of days in which the work has to be completed, decreases, the number of labourers required will increase. So when the days in which the work has to be completed reduce from 10 to 5, the men needed would increase proportionally i.e.labourers would now be needed.

But since the new labourers have twice the efficiency, only half of them will be needed i.e. 18 laborers.

Taking yet another example, let's establish a method of doing these kind of questions:

E.g. 3: If 6 painters can paint 18 walls in 2 week, how many walls of twice the area can 16 painters paint in three weeks?

We are supposed to find the number of walls. Earlier 14 walls were painted. So start by writing '18 × ' on your sheet.

Now the number of painters has increased from 6 to 16. Thus more walls would be painted. Which of the ratio, $\frac{16}{6}$ or $\frac{6}{16}$, when multiplied with 18 will lead to more number of walls than 18? Obviously only when you multiply by a number greater than 1, will the number of walls increase, so it has to be $\frac{16}{6}$.

Hence since painters have increased from 6 to 15, the number of walls painted will be $18 \times \frac{16}{6}$.

Don't calculate it as yet.

Next since the walls are twice the size, the number of walls painted will be less, to be precise will be $\frac{1}{2}$ of earlier i.e. $18 \times \frac{16}{6} \times \frac{1}{2}$.

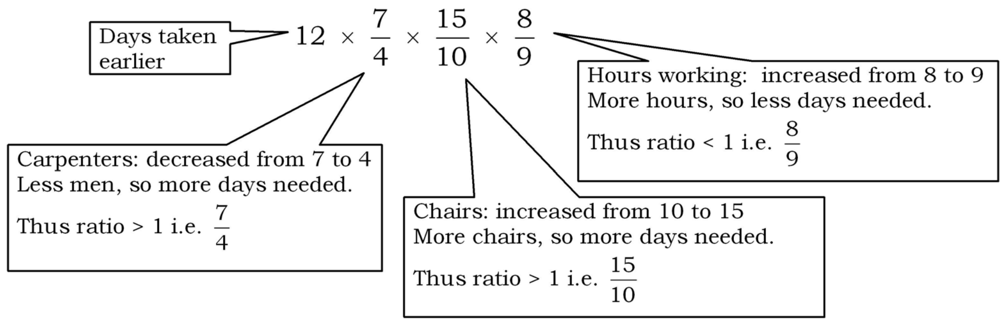
Lastly since the number of weeks has increase from 2 to 3, one will be able to paint more walls. 'More' suggests a ratio greater than 1 i.e. $\frac{3}{2}$. Thus, the number of walls that can be painted is

$18 \times \frac{16}{6} \times \frac{1}{2} \times \frac{3}{2}$ i.e. 36 walls.

E.g. 4: If 8 carpenters build 10 chairs in 12 days working 9 hours a day, how many days will it take for 4 carpenters to build 15 chairs if they work for 8 hours a day?

In the above question we need to find the number of days. So start with the number of days and then taking each of the other changing variable one at a time, just spend a thought whether the number of days will increase or decrease because of the change. E.g. if number of carpenters decrease, number of days required to build the same number of chairs will be more. Also if number of chairs to be made increases, the number of days required to make them will also be more.

Accordingly multiply the number of days with the ratio of the variables that are changing i.e. number of carpenters and number of chairs here. Multiply with a ratio greater than 1 if the number of days has to increase and with the ratio less than 1 if the number of days has to decrease.



Thus, number of days needed = $12 \times \frac{7}{4} \times \frac{15}{10} \times \frac{8}{9} = 28$.

Concept of man-days or man-hours

Another way of approaching such problems is by quantifying the work as man-days or man-hours. E.g. If 8 carpenters build 10 chairs in 12 days working 9 hours a day, then the work i.e. 10 chairs can be quantified as involving 8 men × 12 days × 9 hrs/day i.e. 8 × 12 × 9 man-hours.

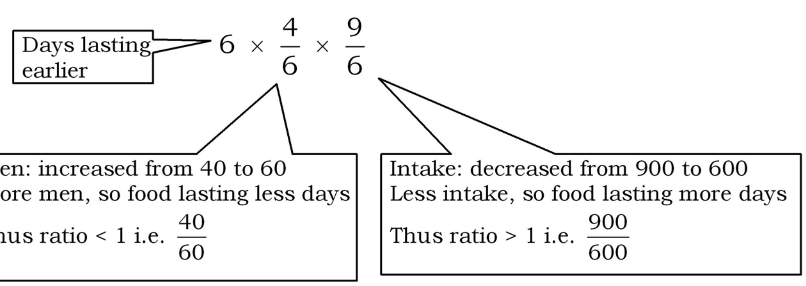
Needless to add, the work and required man-hours are directly proportional (if work doubles, the man-hours needed will double). Thus, if n, d, h, e and w refer to the number of men, days needed, hours per day, efficiency of workers and work respectively, then we can also use a formulaic approach to solve using

$$\frac{n_1 \times d_1 \times h_1 \times e_1}{w_1} = \frac{n_2 \times d_2 \times h_2 \times e_2}{w_2}$$

The question may not have all of n, d, h and e. If some of the variables are not relevant to the question, they can be ignored.

E.g. 5: A garrison has enough food to sustain its 40 men for 6 months if each man consumes 900 gm of food per day. A reinforcement of 20 men join the garrison. For how long will the food last now, if each man now consumes only 600 gm of food per day?

Approach 1: Using proportionality directly



Thus the food will again last for 6 months.

Approach 2: Using man-days

One could also quantify the amount of food (equivalent of work in this example) in terms of man-days.

'enough food to sustain its 40 men for 6 months if each man consumes 900 gm of food per day' means that the food available is equivalent of 40 × 6 × 900 man-month-grams

Now when 20 more men join in, the number of men becomes 60 and since each eat 600 gm food each day, the available quantity of food will last for $\frac{40 \times 6 \times 900 \text{ man-month-grams}}{60 \times 600 \text{ man-grams}}$ = 6 months.

E.g. 6: Eight men working 10 hrs everyday can completely build a wall of length 180 m, breadth 4 m and height 20 m in 20 days. In how many days can 12 men working 8 hrs a day build a wall of length 300 m, breadth 6 m and height 12 m?

Since we need to find the number of days required, we start with the number of days taken ini the earlier instance and proceed as ...

Approach 1: Using proportionality directly

$$\text{Number of days needed} = 20 \times \underbrace{\frac{8}{12}}_{\substack{\text{8 men}\to\text{12 men} \\ \text{more men} \Rightarrow \\ \text{less days}}} \times \underbrace{\frac{10}{8}}_{\substack{\text{10 hrs}\to\text{8 hrs} \\ \text{less hours} \Rightarrow \\ \text{more days}}} \times \underbrace{\frac{300}{180}}_{\substack{\text{180 m}\to\text{300 m} \\ \text{more length} \Rightarrow \\ \text{more days}}} \times \underbrace{\frac{6}{4}}_{\substack{\text{4 m}\to\text{6 m} \\ \text{more breadth} \Rightarrow \\ \text{more days}}} \times \underbrace{\frac{12}{20}}_{\substack{\text{20 m}\to\text{12 m} \\ \text{less height} \Rightarrow \\ \text{less days}}} = 25 \text{ days}$$

Approach 2: Using man-days

Total volume of the wall that is build in 8 × 10 × 20 man-hours = 180 × 4 × 20 m^3

So in 1 man-hour, the volume of the wall that is built = $\frac{180 \times 4 \times 20}{8 \times 10 \times 20}$ = 9 m^3

The number of man-hours required to build the required wall = $\frac{300 \times 6 \times 12}{9}$ = 2400 man-hours

Hence number of days required = $\frac{2400}{12 \times 8}$ = 25 days

Approach 3: Using the formula $\frac{n_1 \times d_1 \times h_1}{w_1} = \frac{n_2 \times d_2 \times h_2}{w_2}$

$$\frac{8 \times 20 \times 10}{180 \times 4 \times 20} = \frac{12 \times d \times 8}{300 \times 6 \times 12}$$

Cancelling out and cross multiplying will result in d = 25

Exercise

1. If 10 cows can eat 10 bags of grain in 10 days, how many days will it take for 1 cow to eat 1 bag of grain?

   a. $\frac{1}{100}$ b. $\frac{1}{10}$ c. 1 d. 10

2. If 17 labourers can dig a ditch 20 m long in 18 days, working 8 hours a day, how many more labourers should be engaged to dig a similar ditch 39 m long in 6 days, each labourer working 9 hours per day.

   a. 89 b. 84 c. 78 d. 72

3. If 30 men working 7 hours a day can do a piece of work in 18 days, in how many days will 21 men working 8 hours a day do a work that is twice as large as the first work.

   a. 15 b. 30 c. 45 d. 60

4. If 30 men complete $\frac{1}{7}^{th}$ the work in 2 days, how many more days will be taken by 30 men to complete the remaining work?

   a. 8 b. 10 c. 12 d. 14

5. 10 students can do a job in 8 days but on the starting day two of them informed that they are not coming. By what fraction will the number of days required for doing the whole work get increased.

   a. 2 b. 1 c. 1/2 d. 1/4

6. If 18 pumps can raise 2170 tonnes of water in 10 days working 7 hours a day, how many days will be required by 16 pumps to raise 1736 tonnes of water working 9 hours a day.

   a. 7 b. 8 c. 9 d. 10

7. If 5 engines consume 6 metric tonnes of coal when each is running for 9 hours a day, how many metric tonnes of coal will be needed for 8 engines, each running 10 hours a day, given that 3 engines of the former type consume as much as 4 engines of the latter type.

   a. 8 b. 9 c. 12 c. 14

8. Some persons can do a piece of work in 12 days. Twice the number of people can finish half the work in how many days?

   a. 3 b. 4 c. 6 d. 24

9. A certain number of men can finish a piece of work in 100 days. If there were 10 men less, it would take 10 days more for the work to finish. How many men were there originally?

   a. 90 b. 100 c. 110 d. 120

10. In a camp, 95 men had provisions for 200 days. After 5 days, 30 men left the camp. How many more days will the remaining food last?

    a. 200 b. 250 c. 285 d. 300

11. A wheel that has 6 cogs is meshed with a larger wheel of 14 cogs. When the smaller wheel has made 21 revolutions, then find the number of revolutions made by the larger wheel.

    a. 27/7 b. 9 c. 49 d. 343/3

12. A rope makes 70 rounds of the circumference of a cylinder whose radius of the base is 14 cm. How many times can the rope go around a cylinder with radius of base being 20 cm.

    a. 49 b. 100 c. 34.3 d. 142.8

Direct & Inverse Variation

Direct Variation

Speaking in general terms, two variables, x and y, are said to vary directly if when one of them increases, the other also increases **proportionally** and if one of them decreases, the other also decreases **proportionally**.

The word 'proportionally' is important here. If one of them doubles, the other should also double; if one of them becomes one-third, the other should also become one-third.

E.g. Speed and Distance covered in same time, say 1 hr, behave in the above manner. Consider I cover a certain distance at a certain speed. If the speed doubles, the distance covered also doubles and if the speed becomes one-third, the distance covered also becomes one-third.

If two variables, x and y vary directly, then the relation is depicted as $x \alpha y$.

As the variable x assumes different values, say x_1, x_2, x_3, …, the variable y will also change and let's say it assumes y_1, y_2, y_3, ... values correspondingly.

E.g. The variables could be 'the number of men working' and 'the number of chairs made'. Now from common sense we know that the number of chairs build and the number of men working are directly proportional (number of men increasing means the number of chairs build also increases proportionally). Next the number of men working could vary i.e. the number of men working could be 5 or 8 or 10 or 25. Accordingly the number of chairs build would also keep changing. Thus, if number of men is denoted as x_1, x_2, x_3, x_4, in the different scenarios, the number of chairs build in the respective scenarios would be denoted as y_1, y_2, y_3, y_4. Thus, each instance of a certain number of men working and the corresponding number of chairs made will be a pair like (x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4).

In the case of direct relation, $x \alpha y$, we would have the relation $\frac{x_1}{y_1} = \frac{x_2}{y_2} = \frac{x_3}{y_3} = \frac{x_4}{y_4} = ...$.

Alternately, $\frac{x}{y} = k$ or $x = ky$, where k is a constant i.e. the pair (x, y) could be any of (x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4) but for any pair, the constant k, will be the same.

In questions we would typically have only two instances of the variables, x and y, assuming different values e.g. in the question 'if 20 people can build 30 chairs, find the number of chairs that 50 people can build?', there are two instances of the data – instance 1: 20 people, 30 chairs; instance 2: 50 people, x chairs.

One can use any of the following approaches:

Approach 1: Use ratios being equal i.e. $\frac{x_1}{y_1} = \frac{x_2}{y_2}$ or $\frac{x_1}{x_2} = \frac{y_1}{y_2}$ (both are the same, use any relation that one happens to write then).

Thus, in this case $\frac{20}{30} = \frac{50}{?} \Rightarrow x = 75$. Or one can write the ratios as $\frac{20}{50} = \frac{30}{x}$, which again gives $x = 75$.

Approach 2: Use $x = ky$, where k is a constant.

One set of corresponding values (x, y) will be given – 20 people, 30 chairs. Use these values to find the value of the constant k. i.e. $30 = 20k$ i.e. $k = 1.5$

Then use the relation again with, with the already found value of k, to find the unknown in the other set of data i.e. $x = 50 \times 1.5$ i.e. $x = 75$.

E.g. 1: y varies directly as x and when $x = 6$, $y = 24$. What is the value of y, when $x = 5$?

Approach 1: Using $\frac{x_1}{y_1} = \frac{x_2}{y_2}$

We have $\frac{6}{24} = \frac{5}{y_2} \Rightarrow y_2 = 5 \times 4 = 20$.

Approach 2: Using $x = ky$

From the set of given values of (x, y), we have $6 = k \times 24$ i.e. $k = \frac{1}{4}$

Using this value of k in the other set of data $(5, y)$, we have $5 = \frac{1}{4} \times y \Rightarrow y = 5 \times 4 = 20$.

E.g. 2: The value of a diamond varies directly as the square of its weight. If a diamond weighing 1.5 gms has a value of Rs. 9,000, what will be the value of a diamond weighing 2.5 gms?

Approach 1: Using $\frac{x_1}{y_1} = \frac{x_2}{y_2}$

Denoting value by v and weight by w, in this case we have $\frac{v_1}{w_1^2} = \frac{v_2}{w_2^2}$

Plugging values, we get $\frac{9000}{1.5^2} = \frac{v_2}{2.5^2} \Rightarrow v_2 = \frac{6.25}{2.25} \times 9000 = \frac{25}{9} \times 9000 = 25{,}000$.

Approach 2: Using $x = ky$

Since the value of the diamond is directly proportional to square of weight, we will have $v = k \times w^2$

From the set of given values of (v, w), we have $9000 = k \times 1.5^2$ i.e. $k = \frac{9000}{2.25} = \frac{9000 \times 100}{9 \times 25} =$ 4,000.

Using this value of k in the other set of data $(v_2, 2.5)$, we have $v_2 = 4000 \times 2.5^2 = 4000 \times 6.25 =$ 25,000.

E.g. 3: The volume of a sphere is directly proportional to the cube of its radius. If the volumes of two spheres are in the ratio 8 : 1, find the ratio of the radii of the spheres.

Representing volume of the sphere by 'v' and radius by 'r', based on the given proportionality relation we would have $\frac{v_1}{v_2} = \frac{r_1^3}{r_2^3}$ i.e. $\frac{v_1}{v_2} = \left(\frac{r_1}{r_2}\right)^3$. (We could also have written the relation as follows $\frac{v_1}{r_1^3} = \frac{v_2}{r_2^3}$, which when rearranged to result in v_1/v_2, given in the question, would have resulted in the same expression)

We are given that $\frac{v_1}{v_2} = \frac{8}{1}$. Thus, $\left(\frac{r_1}{r_2}\right)^3 = \frac{8}{1} \Rightarrow \frac{r_1}{r_2} = \frac{2}{1}$.

Inverse Variation

Speaking in general terms, two variables, x and y, are said to vary inversely if when one of them increases, the other decreases **proportionally** and if one of them decreases, the other increases **proportionally**.

The word 'proportionally' is important here. If one of them doubles, the other should halve; if one of them becomes one-third, the other should become thrice.

E.g. Speed and Time taken to cover a certain distance behave in the above manner. Consider I take a certain time to go from home to office at a certain speed. If the speed doubles, the time taken will halve and if the speed becomes one-third, the time taken would become thrice.

If two variables, x and y vary inversely, then the relation is depicted as $x \; \alpha \; \frac{1}{y}$.

As the variable x assumes different values, say x_1, x_2, the variable y will also change and let's say it correspondingly assumes values y_1, y_2. By now you would be clear by the meaning of x_1, x_2, y_1 and y_2.

In the case of inverse relation, we would have the relation $x_1 \times y_1 = x_2 \times y_2$.

Alternately, we could use the relation, $x = \frac{k}{y}$ or $x \times y = k$, where k is a constant i.e. the pair (x, y) could be any of (x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4), ... but for any pair, the constant k, will be the same.

In questions we would typically have only two instances of the variables, x and y, assuming different values e.g. in the question 'if a wall can be build in 10 days by 18 people, how many people will be needed to build the wall in 15 days?', the variables are 'number of people' and 'days taken' and they vary inversely. Further there are two instances of this set of data – instance 1: 18 people, 10 days; instance 2: x people, 15 days.

One can use any of the following approaches:

Approach 1: Use $x_1 \times y_1 = x_2 \times y_2$

Thus, in this case $18 \times 10 = 15 \times x$ i.e. $x = 12$.

Approach 2: Use $x = \frac{k}{y}$, where k is a constant.

One set of corresponding values (x, y) will be given – 18 people, 10 chairs. Use these values to find the value of the constant k.

$18 = \frac{k}{10}$ i.e. $k = 180$

Then use the relation again with, with the already found value of k, to find the unknown in the other set of data.

$x = \frac{180}{15}$ i.e. $x = 12$.

E.g. 4: If y varies inversely as x, when $x = 2$, $y = 3$. What is the value of x if $y = 1$?

Approach 1: Use $x_1 \times y_1 = x_2 \times y_2$

Using this relation we will have $2 \times 3 = x \times 1$ i.e. $x = 6$.

Approach 2: Use $x = \frac{k}{y}$ or $x \times y = k$, where k is a constant.

From the set of given values of (x, y), we have $2 \times 3 = k$ i.e. $k = 6$.

Using this value of k in the other set of data $(x, 1)$, we have $x \times 1 = 6$ i.e. $x = 6$.

E.g. 5: The number of four wheelers sold in a year vary inversely with the number of two wheelers sold in that year. If 4000 four wheelers were sold in one particular year, 21000 two wheelers were sold that year. How many two wheelers were sold in a year, when 6000 four wheelers were sold?

Approach 1: Use $x_1 \times y_1 = x_2 \times y_2$

Using this relation we will have 4,000 × 21,000 = 6,000 × y i.e. y = 14,000.

Approach 2: Use $x = \frac{k}{y}$ or $x \times y = k$, where k is a constant.

From the set of given values of (4000, 21000), we have 4000 × 21000 = k i.e. k = 84,000,000.

Using this value of k in the other set of data (6000, y), we have 6000 × y = 84,000,000 i.e. y = 14,000.

E.g. 6: The cost per kg of rice varies inversely with the square of the quantity of rice produced in a year. When 7 million tons of rice was produced, its cost was Rs. 36/kg. How much was the production in the year when the cost of rice was Rs. 49/kg?

Denoting price by p and quantity by q, the relation in this case will be $p_1 \times q_1^2 = p_2 \times q_2^2$.

Plugging the given values, 36 × 72 = 49 × q_2^2 i.e. $q_2 = 6$.

Joint Variation

Sometimes more than two variables could be involved in the questions, say x, y and z and the proportionality relation is such that x is directly proportional to y and x is inversely proportional to z.

Since x is common to the two proportionality relation, we should form two separate relations $\frac{x}{y}$ = some constant and $x \times z$ = another constant. And then the two relation can be clubbed into one relation like $\frac{x \times z}{y} = k$. From this last relation, for difference instances of the three variables, (x_1, y_1, z_1) and (x_2, y_2, z_2), we will have $\frac{x_1 \times z_1}{y_1} = \frac{x_2 \times z_2}{y_2}$.

E.g. 7: The value of a circular coin varies directly with the square of it’s radius and varies inversely with it’s thickness. What is the ratio of values of two coins with the ratio of their radii being 2 : 3 and the ratio of their thickness being 1 : 4.

If the value, radius and thickness is denoted by v, r and t respectively, using the proportionality relation we have $\frac{v}{r^2}$ = a constant and $v \times t$ = a constant.

Clubbing the two relations together we have $\frac{v \times t}{r^2}$ = a constant.

Now we have two coins, the ratio of who’s values have to be found i.e. we need to find v_1/v_2. And we are given that $\frac{r_1}{r_2} = \frac{2}{3}$ and $\frac{t_1}{t_2} = \frac{1}{4}$.

From the relation tying up the three variables, we have $\frac{v_1 \times t_1}{r_1^2} = \frac{v_2 \times t_2}{r_2^2}$. On re-arranging we have $\frac{v_1}{v_2} = \left(\frac{r_1}{r_2}\right)^2 \times \frac{t_2}{t_1}$. Plugging the given data, we find, $\frac{v_1}{v_2} = \left(\frac{2}{3}\right)^2 \times \frac{4}{1} = \frac{16}{9}$.

Exercise

1. y varies directly as the square of x. When $x = 4$, $y = 2$. What is the value of y when $x = \frac{3}{2}$?

   a. $\frac{9}{32}$ b. $\frac{3}{4}$ c. $\frac{16}{3}$ d. $\frac{128}{9}$

2. y varies inversely as the cube root of x. If $y = 7$, when $x = 8$, what is the value of x when $y = 56$?

   a. 1/2 b. 1/8 c. 1/3 d. 1/64

3. The value of a bar of gold weighing 400 gm, varies inversely with the square root of the fraction of impurities in it. If the value of a bar containing 25 gms of impurities is Rs. 45000, how many gms of pure gold is there in a bar costing Rs. 90000?

   a. 12.5 b. 387.5 c. 375 d. 393.75

4. The value of a silver coin varies directly as the square of its diameter, when thickness is constant and varies directly as its thickness when diameter remains constant. Two silver coins have the diameters in the ratio 4 : 3. Find the ratio of the thickness if the value of the first coin is four times the value of the second coin.

   a. 3 : 4 b. 4 : 3 c. 16 : 3 d. 9 : 4

5. The value of a diamond varies directly as the square of its weight. If a diamond falls and breaks into two pieces with weights in the ratio 2 : 3, what is the loss percentage in the value?

   a. No loss b. 16.66% c. 25% d. 48%

Assignment: Ratio Proportion Variation

1. $\frac{A}{B}=\frac{3}{2}$ and $\frac{B}{4}=\frac{C}{3}$. Find the ratio A : C.

   (a) 1 : 2 (b) 2 : 1 (c) 1 : 1 (d) 3 : 4

2. If $\frac{x}{y}=\frac{3}{4}$, find the value of $\frac{2x^2+3y^2}{2x^2-3y^2}$

   (a) $-\frac{11}{5}$ (b) $\frac{11}{5}$ (c) $\frac{16}{7}$ (d) $\frac{1}{4}$

3. 5/9 to what fraction bears the same ratio as 1/27 to 11/3?

   (a) 1/55 (b) 1/11 (c) 3/11 (d) 55

4. A sum of Rs.1900 is to be divided among A, B and C such that $\frac{A}{B}=\frac{B}{C}=\frac{2}{3}$, where A, B and C are their respective shares. What is the share of B and C put together?

   (a) 1400 (b) 1500 (c) 1600 (d) 800

5. A is the mean proportion of 36 and 49 and D is fourth proportion of 2, 7, 6. Find A : D

   (a) 2 (b) 2/3 (c) 4/7 (d) 6/7

6. If $2a = 3b = 5c$ and $a + c - b = 1331$, then $\frac{a}{5}+\frac{b}{2}+\frac{2c}{3}=$

   (a) 1210 (b) 1441 (c) 1452 (d) 1456

7. If $\frac{x}{2}=\frac{y}{3}=\frac{z}{5}=\frac{x+y-z}{k}$, find the value of k.

   (a) –1 (b) 0 (c) 1 (d) 2

8. If ages of A and B, five years back, were in the ratio 4 : 1 and fifteen years from now they will be in the ratio 2 : 1, then their present ages are:

   (a) 30, 10 (b) 45, 15 (c) 21, 7 (d) 60, 20

9. An amount of 2430 is to be distributed among A, B and C such that if their shares are reduced by 5, 10 and 15 respectively, the residual amounts are in the ratio 3 : 4 : 5. Then B's share is:

   (a) 605 (b) 790 (c) 800 (d) 810

10. Ratio of a two digit number to the sum of its digits is 4. Ratio of the tens digit to the unit digit of the number is

    (a) 2/3 (b) 4/5 (c) 1/2 (d) more than one value

11. The ratio of incomes of A and B is 3 : 2 and the ratio of expenses of A and B are in the ratio 2 : 3. If they save Rs. 200 and Rs. 100 respectively, find the income of A.

    (a) 240 (b) 300 (c) 360 (d) 480

12. To make a furniture three different type of parts, A, B and C, are required. The ratio of cost of each unit of A, B and C is 2 : 3 : 6. The respective quantities required of these parts are in ratio 3 : 2 : 1. What fraction of the cost of the furniture is accounted by part B?

(a) 1/3 (b) 3/11 (c) 1/2 (d) Cannot be determined

13. In a partnership, if A invests Rs. 2000 and B invests Rs. 3000 and if the total profit of the partnership is Rs.1500, what is difference between the shares of A and B?

(a) 250 (b) 300 (c) 500 (d) 350

14. A business is started by two people, A and B, with respective investments of Rs. 1400 and Rs. 2100. A being the working partner takes 25% of the profit as salary. If at the end of a year they make a profit of Rs. 2000, what will be B's share of the profit?

(a) 800 (b) 500 (c) 1500 (d) 900

15. A and B start a partnership with investment in ratio 3 : 2. After six months B doubles his investment. In what ratio will they distribute the profit at the end of the year?

(a) 3:4 (b) 2:3 (c) 1:1 (d) 6:4

16. For a social work campaign, which required 200 volunteers to clean 50 villages, if 16 volunteers did not turn up, how many villages can be are cleaned now?

(a) 40 (b) 34 (c) 46 (d) 42

17. If 20 carpenters build 10 chairs in 5 days, how many days will be needed by 5 carpenters to make 20 chairs?

(a) 10 (b) 20 (c) 40 (d) 80

18. A garrison of 60 soldiers had sufficient food to last them for 80 days. A reinforcement of 40 soldiers joins the garrison. The daily ration to each soldier is reduced to three-fourth of earlier ration. How many days will the food last now?

(a) 50 (b) 60 (c) 64 (d) 75

19. The volume of a cylinder varies directly with the square of the radius of the base and also varies directly with the height of the cylinder. Two cylinders having the ratio of radii of base as 2 : 1 have equal volumes. What is the ratio of heights of the two cylinders?

(a) 1 : 1 (b) 1 : 2 (c) 1 : 4 (d) 1 : 6

20. Price of a diamond varies directly as square of its weight. A diamond of 2.5 gms costs Rs. 80,000. What will be the weight of a diamond costing Rs. 3,20,000?

(a) 4 gms (b) 5 gms (c) 7.5 gms (d) 10 gms

Percentages

Finding percent equivalent of '*a* out of *b*'

Percent literally means 'per cent', cent as in century. Thus, percent means 'per 100' and is represented with the symbol '%'.

This means that any data, *a* out of *b*, is proportionally reduced or increased so as to be 'out of 100'

E.g. A score of 80 marks out of 150 marks. To find the percentage of marks scored, we need to find the equivalent marks scored had the maximum marks been 100

80 marks out of 150

$\Rightarrow \frac{80}{150}$ marks out of 1 (Dividing by 150)

$\Rightarrow \frac{80}{150} \times 100$ marks out of 100. (Multiplying by 100)

Thus, 80 marks out of 150 is equivalent to a percent of $\frac{80}{150} \times 100$ i.e. 53.33%

E.g. In a class of 60 students, 45 of them are girls. To find the percent of girls in the class, we have to find the equivalent number of girls had the strength of the class been 100.

45 girls in a class of 60

$\Rightarrow \frac{45}{60}$ girls in a class of 1 (Dividing by 60)

$\Rightarrow \frac{45}{60} \times 100$ girls in a class of 100 (Multiplying by 100)

To convert the data '*a* out of *b*' to a percentage .... we need to do the following: $\frac{a}{b} \times 100$

One needs to be slightly careful of what comes in the denominator. It is always the data associated with 'of' – 80 marks out **<u>of 150</u>** or 45 girls in a class **<u>of 60</u>**

E.g. 1: 50 is what percent of 80? Also find 80 as a percentage of 50.

As seen in the above examples, the required answer is $\frac{50}{80} \times 100 = \frac{250}{4}$ = 62.5%

While finding 80 as a percentage of 50, we would need to do: $\frac{80}{50} \times 100 = 160\%$

Finding *x*% of *y*

A percent figure is always a percentage **<u>of</u>** something. Thus, when we say the attendance was 80%, we mean that 80% of the total students were present; when we say that Rohit scored 85%, we mean 85% of the maximum marks; when we say that a number increased by 20%, we mean 20% of itself; when we say the profit was 40%, we mean 40% of the cost; when we say the sales grew by 35%, we mean 35% of the earlier sales figure.

To find x% of y ... ... we need to do $\frac{x}{100} \times y$

Exercise

1. Find 0.25% of 40,000

   a. 100 b. 1,000 c. 10 d. 10,000

2. 32 is what percentage of 44?

   a. 70% b. 88.88% c. 80% d. 72.72%

3. 175 is what percentage of 125?

   a. 140% b. 40% c. 71.42% d. 128.56%

4. When 30 is subtracted from a number, the result is 80% of the number. Find the number.

   a. 120 b. 110 c. 150 d. 80

5. Mohan spends 40% of his income on rent, 20% on food, 10% on entertainment and saves the rest. If he saves a net amount of Rs. 4380, find his income.

   a. Rs. 21,900 b. Rs. 14,600 c. Rs. 1,46,000 d. Rs. 29,200

6. In an election between two candidates, the first candidate got 60% of the votes polled and won by a majority of 25,000 votes over the second candidate. Find the total number of votes polled.

   a. 125,000 b. 62,500 c. 105,000 d. 250,000

7. In a mixture of milk and water, milk accounts for 75%. Find the amount of the mixture if it contains 28 lts of water.

   a. 42 lts b. 49 lts c. 112 lts d. 40 lts

8. A milk-man mixes milk and water in the ratio 5 : 3. Find the percentage of milk in the resulting mixture.

   a. 48% b. 37.5% c. 62.5% d. 52%

9. When 1kg of goods is kept on an electronic weighing scale, the scale shows a reading of 950 gms. Find the percentage error in the measurement.

   a. 5.26% b. 3% c. 4.8% d. 5%

10. A unscrupulous trader has rigged his electronic weighing scale such that it shows a reading of 1 kg when 950 gms of goods are kept on it. Find the percentage error in the measurement.

    a. 4% b. 5.26% c. 5% d. 4.5%

Percentage-fraction equivalence

In the course of solving questions on percentages, we would very often have to find the percentages from expressions like $\frac{5}{6}\times 100, \frac{10}{11}\times 100, \frac{5}{8}\times 100$ or we would be finding 37.5% or 22.22% or 18.18% of a certain number. These could be tedious calculations and unnecessarily wear out a student. On the contrary these are very easy if one has already learnt a few percentage-fraction equivalence.

E.g. $\frac{2}{11}$ as a fraction is equivalent to 18.18%. This means that $\frac{2}{11}\times 100 = 18.18\%$ and 18.18% of n is same as $\frac{2}{11}\times n$

E.g. Knowing that $37.5\% \approx \frac{3}{8}^{th}$, the next time we have to find $\frac{3}{8}\times 100$, one can write the answer in a jiffy as 37.5%.

This will also help us in finding 37.5% of say 72. Rather than perform the tedious calculation $\frac{37.5}{100}\times 72$, one could just orally think of the expression as $\frac{3}{8}\times 72$ to arrive at 27 as the answer.

The following is a comprehensive list of the percentage-fraction equivalence that you would need to memorise. Groups that can be memorized together are given in one row.

$\frac{1}{2} = 50\%,\ \frac{1}{4} = 25\%,\ \frac{1}{8} = 12.5\%,\ \frac{1}{16} = 6.25\%$

$\frac{1}{3} = 33.33\%,\ \frac{2}{3} = 66.66\%$ $\frac{1}{6} = 16.66\%,\ \frac{5}{6} = 83.66\%$ $\frac{1}{12} = 8.33\%$

$\frac{1}{4} = 25\%,\ \frac{3}{4} = 75\%$

$\frac{1}{5} = 20\%,\ \frac{2}{5} = 40\%,\ \frac{3}{5} = 60\%,\ \frac{4}{5} = 80\%$

$\frac{1}{7} = 14.28\%,\ \frac{2}{7} = 28.56\%,$ $\frac{1}{14} = 7.14\%$

$\frac{1}{8} = 12.5\%,\ \frac{3}{8} = 37.5\%,\ \frac{5}{8} = 62.5\%,\ \frac{7}{8} = 87.5\%$

$\frac{1}{9} = 11.11\%,\ \frac{2}{9} = 22.22\%,\ \frac{4}{9} = 44.44\%,\ ......,\frac{7}{9} = 77.77\%$

$\frac{1}{11} = 9.0909\%,\ \frac{2}{11} = 18.1818\%,\ \frac{3}{11} = 27.2727\%,\ ......,\frac{7}{11} = 63.6363\%,\ ......,\ \frac{10}{11} = 90.9090\%$

$\frac{1}{15} = 6.66\%,\ \frac{1}{16} = 6.25\%$

$\frac{1}{19} = 5.76\%,\ \frac{1}{20} = 5\%,\ \frac{1}{21} = 4.76\%$

Practice

Practice the following to be swift in calculations:

1. Calculate the following percentages:

   a. 16.66% of 48 b. 12.5% of 180 c. 27.27% of 165 d. 44.44% of 63

   e. 6.25% of 24 f. 6.66% of 57 g. 37.5% of 72 h. 83.33% of 42

   i. 87.5% of 144

2. For each of the following find A as a percentage of B i.e. $\frac{A}{B} \times 100$

   a. A = 24, B = 288 b. A = 35, B = 56 c. A = 42, B = 66 d. A = 63, B = 45

   e. A = 20, B = 36 f. A = 88, B = 64

Answers to fraction percentage equivalence

1. a. 8 b. 22.5 c. 45 d. 28 e. 1.5 f. 3.8 g. 27 h. 35 i. 84

2. a. 8.33% b. 62.5% c. 63.63% d. 140% e. 55.55% f. 137.5%

Concept of a base

Whenever a percentage is expressed, it always means a percent of 'some numeric quantity'. E.g. 60% of the maximum marks, 10% of the total votes polled, 60% of the valid votes, 75% of the mixture, prices dropped by 15% (of itself). This 'some numeric quantity' is called the base and it is extremely important to identify the base correctly.

Let's say we want to find 30 is what percent of 45. What is the base in this example? Is it 30 or 45? To be sure of identifying the correct base try thinking on the following lines: do we want to find percentage of 30 or 45? The question makes it very apparent that we have to find 'percent of 45'. Thus our base is 45 and the question can be transformed mathematically to: 30 = ? % × 45.

Since our base is 45, we would have to proportionally convert '30 on a base of 45' into 'x on a base of 100'. The answer can be found directly without writing any of the above as $\frac{30}{45} \times 100 = \frac{2}{3} \times 100 = 66.66\%$.

Had the question been '45 is what percent of 30?', our base would be 30 and to find the answer we would have to proportionally convert '45 on a base of 30' into 'x on a base of 100'. Thus the answer would have been $\frac{45}{30} \times 100 = \frac{3}{2} \times 100 = 150\%$.

The most common error that a beginner commits is that of incorrectly identifying the base and then finding the percent of this incorrect base. To avoid this mistake always ask the question 'percent of what'.

Compare the following examples of similar situations, one straightforward and other very similar one but in which beginners commit errors. In the second example, the error is made in indentifying the base incorrectly.

I.a. What is 20% of 450?

Here 20% of 450 is to be found, so 450 is your base of which 20% is to be found. Thus the answer is
$\frac{20}{100} \times 450 = 90$

I.b. 20% of what number is 450?

Here you do not have to find 20% of 450. The number of which 20% is to be found is unknown. So if we assume the number to be n, the working would go as follows: $\frac{20}{100} \times n = 450 \Rightarrow n = 2250$

In I.a the base is 450 and in I.b. the base is the unknown number to be found.

II.a. What is the result when 100 is increased by 20 %?

This should be a sitter. Here 100 is increased by 20 %, so we will find 20% of 100 and add it to 100. Thus the answer is 120.

II.b. A number increased by 20% becomes 100. Find the number.

The error is committed because of the presence of 100 and in haste most students find 20% of 100 as 20. And since the number has to be less than 100, the number is found as 100 – 20 = 80.

Please notice that 20% is of the number and not of 100. 100 is the final result after an increase of 100. Thus the correct approach should be assuming the number as n and then forming the equation
$n + \frac{20}{100} \times n = 100 \Rightarrow \frac{6}{5} n = 100 \Rightarrow n = \frac{500}{6} = 83.333$

In II.a the base is 100 and in II.b. the base is the unknown number to be found.

E.g. 2: In a poll, a total of 1,20,000 votes were polled of which 10% were invalid. A candidate received 60% of the valid votes. Find the number of votes polled by the candidate.

The number of invalid votes = 10% of 1,20,000 = $\frac{10}{100} \times 1,20,000$ = 12,000.

Thus, the number of valid votes = 1,20,000 – 12,000 = 1,08,000.

The candidate received 60% of 1,08,000 i.e. $\frac{60}{100} \times 1,08,000$ = 64,800.

E.g. 3: When 40% of a number is subtracted from 100, the answer is 80% of the number. Find the number.

If the number is n, then $100 - \frac{40}{100} \times n = \frac{80}{100} \times n$ i.e. $100 - 0.4n = 0.8n$ i.e. $1.2n = 100$ i.e. $n =$ 83.33

E.g. 4: 100 is 30% of what number?

If the number is assumed to be n, $\frac{30}{100} \times n = 100 \Rightarrow n = \frac{1000}{3}$ = 333.33

E.g. 5: Ram's salary is 90% of Shyam's salary. Shyam's salary is what percent of Ram's salary?

R is 90% **of S**. Thus, $\frac{R}{S} \times 100 = 90 \Rightarrow \frac{R}{S} = \frac{9}{10}$

The questions requires us to find S as a percent of R i.e. $\frac{S}{R} \times 100$. And the required answer will be $\frac{10}{9} \times 100 = \frac{1000}{9}$ = 111.11%

Alternately, Ram's salary is 90% that of Shyam's salary So assuming Shyam's salary as 100, we can get Ram's salary as 90% of 100 i.e. 90. Now the question is asking us to find 100 as a percent of 90 i.e. the base now is 90. Thus the answer is $\frac{100}{90} \times 100 = \frac{1000}{9}$ =111.11%

E.g. 6: An alloy consists of 80% Bronze and rest Brass. Bronze and Brass, themselves are alloys with Bronze having 60% Copper and the rest Zinc whereas Brass has Copper and Zinc in equal proportions. What percentage of the alloy of Bronze and Brass is Zinc?

Let's assume the alloy of Bronze and Brass as 100 gms. Thus it will have 80 gms of Bronze and 20 gms of Brass. Let's find the amount of zinc in both. In 80 gms of Bronze, amount of Zinc is 40%. This 40% is of the amount of Bronze i.e. of 80 gms. Quantity of Zinc in Bronze = $\frac{40}{100} \times 80$ = 32 gms. Similarly quantity of Zinc in Brass = $\frac{50}{100} \times 20$ = 10 gms.

Thus the total quantity of Zinc in the alloy is 32 + 10 = 42 gms. Now we have to find the percentage of Zinc in the alloy. The alloy is 100 gms and amount of Zinc in it is 42 gms. Thus the percentage of Zinc is 42%.

E.g. 7: Rohit scored 50% of the maximum marks in an exam and yet failed by 12 marks. Had he scored 10% more than what he scored, he would have just got passing marks. Find the maximum marks of the exam.

Let the maximum marks of the paper be x. Thus Raj scored $\frac{50}{100} \times x = \frac{x}{2}$. Now had he scored 10% more than what he scored means we have to calculate 10% of $\frac{x}{2}$ and this will be equal to 12 marks because he just managed to pass with these additional marks. Thus, we have $\frac{10}{100} \times \frac{x}{2} = 12 \Rightarrow x = 240$.

Alternately, if you wanted to avoid equations, 10% of the marks he scored is equal to 12. Thus he scored 120 marks. And this is 50% of the maximum marks. Thus, maximum marks = 240.

Exercise

11. The prices of a car increased by 20%. If the increased price of the car is Rs. 6,00,000, find the original price of the car.

    a. Rs. 5,60,000 b. Rs. 5,20,000 c. Rs. 5,00,000 d. Rs. 7,20,000

12. *A*'s salary is 80% of *B*'s salary. Find what percent of *A*'s salary is *B*'s salary.

    a. 100% b. 125% c. 80% d. 120%

13. To pass an exam of 250 marks, a candidate should obtain 40% marks. Raj scores 10% more than the passing marks. Find the percent of marks scored by Raj.

    a. 50% b. 40% c. 44% d. 44.44%

14. In a country 55.55% of the population are males and rest females. Of the males 60% are literate and of the females 40% are literate. Find the population that are literate, if the population of the country is 900 million.

    a. 460 b. 400 c. 300 d. 160

15. A milk man mixes water equal to 20% of the milk he has. Water now accounts for what percentage of the mixture?

    a. 16.66% b. 20% c. 10% d. 33.33%

16. Mohan spends 40% of his income on rent, 20% on food, 10% on entertainment and saves the rest. Find his savings as a percentage of his expenditure.

    a. 30% b. 42.85% c. 45% d. 20%

17. Anand has won 80% of the games he has played so far in the tournament. His goal is to win 90% of all the games he has to play in the tournament. If he has already played 15 out of the total 50 games that he has to play, what is the maximum number of games he can afford to loose in the remaining games and yet meet his goal?

    a. 5 b. 4 c. 3 d. 2

18. *A*'s salary is 80% of *B*'s salary whereas *B*'s salary is 80% of *C*'s salary. What percentage of *C*'s salary is *A*'s salary?

    a. 80% b. 64% c. 150% d. 70%

Percent Increase/Decrease – The Multiplying Factor

We will see many applications where a number is increased by certain percentage. This section here explains how these percent increase/decrease can be handled very effectively.

Consider that a number n is increased by 22.22%?

It should be obvious if you have done fraction percentage equivalence that we have to find $n+\frac{2}{9}\times n$. The process of finding 22.22% of n i.e. $\frac{2}{9}^{th}$ of n and then adding it back to n can be compressed into one step by considering our answer in the form $n\times\left(1+\frac{2}{9}\right)$ i.e. $n\times\frac{11}{9}$.

Thus increasing n by 22.22% is same as multiplying it by $\frac{11}{9}$. Thus we say that $\frac{11}{9}$ is the 'multiplying factor' corresponding to an 'increase of 22.22%'.

See if you understand finding the multiplying factors as explained below:

Increase of 16.66% is equivalent to multiplying by $\left(1+\frac{1}{6}\right)$ i.e. $\frac{7}{6}$

Increase of 37.5% is equivalent to multiplying by $\left(1+\frac{3}{8}\right)$ i.e. $\frac{11}{8}$

Increase of 125% is equivalent to multiplying by $\left(1+\frac{5}{4}\right)$ i.e. $\frac{9}{4}$

Decrease of 11.11% is equivalent to multiplying by $\left(1-\frac{1}{9}\right)$ i.e. $\frac{8}{9}$

Decrease of 27.27% is equivalent to multiplying by $\left(1-\frac{3}{11}\right)$ i.e. $\frac{8}{11}$

Similarly, one should be able to work in the reverse way also.

$n\times\frac{8}{7}$ is same as n being increased by $\left(\frac{8}{7}-1\right)=\frac{1}{7}$ i.e. 14.28%

$n\times\frac{11}{9}$ is same as n being increased by $\left(\frac{11}{9}-1\right)=\frac{2}{9}$ i.e. 22.22%

$n\times\frac{7}{8}$ is same as n being decreased by $\left(\frac{7}{8}-1\right)=-\frac{1}{8}$ i.e. 12.5% (Negative sign denotes decrease)

$n\times\frac{2}{5}$ is same as n being decreased by $\left(\frac{2}{5}-1\right)=-\frac{3}{5}$ i.e. 60%

Practice

Convert each of the following percentage change into its equivalent multiplying factor

1. n increases by 8.33% $\Rightarrow$ increased value = $n \times \frac{?}{?}$

2. Prices increase by 16.66% $\Rightarrow$ new price = original price $\times \frac{?}{?}$

3. Length of a rectangle decreases by 12.5% $\Rightarrow$ new length = original length $\times \frac{?}{?}$

4. Sales of a company decreases by 36.36% $\Rightarrow$ sales = previous sales $\times \frac{?}{?}$

5. There is a increase of 9.09% in my salary $\Rightarrow$ new salary = original salary $\times \frac{?}{?}$

6. A discount of 30% is given on the prices $\Rightarrow$ Final price = original price $\times \frac{?}{?}$

In each of the following, first choose which of the two words "increased/decreased" is correct in the context given and then replace the "?" b yan appropriate percent value.

7. price × $\frac{8}{9} \Rightarrow$ that prices are increased/decreased by ? %

8. breadth $\times \frac{13}{11} \Rightarrow$ that breadth is increased/decreased by ? %

9. salary $\times \frac{16}{15} \Rightarrow$ that salary is increased/decreased by ? %

10. original quantity $\times \frac{15}{16} \Rightarrow$ that quantity increases/decreases by ?%

11. If $a = b \times \frac{5}{8}$, which of the following is true?

    (1) b is 37.5% less than a
    (2) a is 37.5% less than b
    (3) b is 37.5% more than a
    (4) a is 37.5% more than b

12. If $a \times \frac{8}{5} = b$, which of the following is true?

    (1) b is 60% less than a
    (2) a is 60% less than b
    (3) b is 60% more than a
    (4) a is 60% more than b

13. If $n = m \times \frac{9}{7}$, which of the following is true?

    (1) n is 14.28% more than m
    (2) m is 14.28% more than n
    (3) n is 14.28% less than m
    (4) m is 14.28% less than n

14. If $n \times \frac{7}{9} = m$

    (1) n is 22.22% more than m
    (2) m is 22.22% more than n
    (3) n is 22.22% less than m
    (4) m is 22.22% less than n

Answers to multiplying fractions

1. $\frac{13}{12}$ 2. $\frac{7}{6}$ 3. $\frac{7}{8}$ 4. $\frac{7}{11}$ 5. $\frac{12}{11}$ 6. $\frac{7}{10}$ 7. decrease 11.11% 8. increase, 18.18%

9. increase, 6.66% 10. decrease, 6.25% 11. 2 12. 3 13. 1 14. 4

Percent Increase/Decrease and Ratios:

When a percentage increase or decrease is given in a quantity, one can also very easily find the ratio of the original and new value (without even knowing what the original value was)

If a number n increases by 10%, the new value will be $\frac{11}{10} \times n$.

Thus the ratio of original value and new value is $n : \frac{11}{10} \times n$ i.e. 10 : 11

Consider one more example:

The pries of a car increases by 12.5%

If the original price of the car was 100, the new price is 112.5

If the original price of the car was x, the new price is $\frac{9}{8} \times x$

If the original price of the car was 8, the new price is 9

All of these is same as: the ratio of original price to new price is 8 : 9.

My salary decreased by 20%

If my salary was 100, it became 80

If my salary was x, it became $\frac{4}{5} \times x$

If my salary was 5, it became 4

All of these is same as: the ratio of original and new salary is 5 : 4

Question Type: Change of Base:

E.g. 8: If A is 20% more than B, by what percent is B less than A?

In such questions there are two variables A and B. The data is given as a percentage of one of them (in this case 20% of B), and the question asked is about percentage of the other variable (in this case percent less than A).

There are basically three ways of doing it.

Method 1: Assuming base as 10

If B is assumed as 100, A will be 120. Thus the answer is $\frac{20}{120} = \frac{1}{6} = 16.66\%$.

This method can be generalised as follows:

If A is x% more than B, then B is $\left(\frac{x}{100 + x}\right) \times 100\%$ less than A, and

Please do not memorize. It should be obvious as if B = 100, A = 100 + x and B is x less than 100 + x which when converted to a percent is the expression given.

If A is x% less than B, then B is $\left(\frac{x}{100 - x}\right) \times 100\%$ more than A.

However this method or formula is easy only when the percent given is a very manageable one. If the question had been 'If A is 37.5% more than B, by what percent is B less than A?', the working using this formula would be cumbersome as it would involve finding the $\frac{37.5}{137.5}$.

Method 2: Using multiplying factors.

If A is 20% more than B, we can say $A = \frac{6}{5} \times B$

Re-writing this, with B in terms of A, $B = \frac{5}{6} \times A$

From the last relation we can say that B is $\frac{1}{6}^{th}$ less than A i.e. 16.66% less than A.

This method is more versatile than the previous one and can be used to solve even if the percentages are a little difficult ones, e.g. If A is 37.5% more than B, by what percent is B less than A?

$$A = \frac{11}{8} \times B \quad \Rightarrow \quad B = \frac{8}{11} \times A$$

Thus B is $\frac{3}{11}^{th}$ i.e. 27.27% less than A

Method 3: Using Ratios

This is the most simplest of all methods and it is just a variation of the second method.

If A is 20% more than B, then the ratio of $A : B$ is 6 : 5.

Thus, B is $\frac{1}{6}^{th}$ i.e. 16.66% less than A.

If A is 37.5% more than B, then the ratio of $A : B$ is 11 : 8

Thus, B is $\frac{3}{11}^{th}$ i.e. 27.27% less than A.

Question Type: Percentage changes in relations like $C = A \times B$

The above concept can also be used in any relation of the type $C = A \times B$. Few very commonly found such relations are:

Area = Length × Breadth

Expenditure or Revenue (Sales) = Price per unit × Quantity

Thus, we could have questions of the following type with any of the above relation:

E.g. 9: If the length of a rectangle is increased by 10%, by what percent should the breadth be decreased to maintain area?

Method 1: Using Multiplying Factors:

Let the original length and breadth of the rectangle be l and b respectively.

Thus, area = $l \times b$.

The length changes from l to $\frac{11}{10}l$ and let breadth change from b to $f \times b$.

Since area is a constant, we would have the relation, $l \times b = \left(\frac{11}{10} \times l\right) \times (f \times b)$.

Obviously, in this relation f has to be $\frac{10}{11}$, making the two sides equal i.e. $l \times b = \frac{11}{10} l \times \frac{10}{11} b$.

Thus, the breadth should become $\frac{10}{11}^{th}$ of the original breadth i.e. $\frac{1}{11}^{th}$ less i.e. 9.09% decrease.

Method 2: Using ratios

The ratio of lengths would be 10 : 11. Only when the ratio of breadths would be 11 : 10, would the area be equal, as shown below:

	Original		New
Length	10	:	11
Breadth	11	:	10
Area	110	:	110

Thus if ratio of original and new breadth is 11 : 10, it's a decrease of $\frac{1}{11}^{th}$ i.e. 9.09%

E.g. 10: If prices of a commodity decrease by 8.33%, by what percentage can the consumption (quantity) increase in same expenditure.

Using Multiplying Factors:

Original Price × Original Consumption = $\left(\frac{11}{12} \times \text{original price}\right) \times \left(\frac{12}{11} \times \text{original consumption}\right)$

Thus consumption could increase by $\frac{1}{11}^{th}$ i.e. 9.09%

Using Ratios:

The ratio of prices (original and decreased) is 12 : 11

The ratio of quantity should be 11 : 12, if expenditure has to be constant.

Thus quantity increased by by $\frac{1}{11}^{th}$ i.e. 9.09%

Exercise

19. If A is 16.66% more than B, by what percent is B less than A?

    a. 16.66% b. 20% c. 14.28% d. 10%

20. If M is 22.22% less than N, by what percent is N more than M?

    a. 28% b. 28.56% c. 33.33% d. 20%

21. If X is 22.22% more than Y, by what percent is Y less than X?

    a. 20% b. 22.22% c. 18.18% d. 16.66%

22. If P is 37.5% less than Q, by what percent is Q more than P?

    a. 15% b. 37.5% c. 30% d. 60%

23. If the length of a rectangle increases by 9.09%, by what percent should the breadth of the rectangle decrease to maintain the area of the rectangle?

    a. 11.11% b. 9.09% c. 10% d. 8.33%

24. If the price of the entrance ticket of a circus is decreased by 6.25%, by what percent should the number of viewers increase so as to earn as much as that earned before the decrease in price?

    a. 5% b. 6.25% c. 7.5% d. 6.66%

25. Traveling from home to office, if I increase my speed by 12.5% than my usual speed, what percent of time would I save?

    a. 10% b. 11.11% c. 12.5% d. 15%

26. Prices of essential commodities decrease by 27.27%. By what percent can a household increase its consumption with the same expenditure?

    a. 37.5% b. 30% c. 20% d. 33.33%

Finding actual quantities when percentage increase and an absolute change is given

In all the above questions, we were asked to find the percentage difference in one quantity when a percentage difference is given in another quantity. Questions could also be a step further than this:

E.g. 11: When the length of the rectangle is increased by 10%, the breadth has to be reduced by 3 cms to maintain area. Find the original and new breadth.

Because length increases by 10%, the ratio of the original and new length is 10 : 11.

To maintain the area, the ratio of the original and new breadth is 11 : 10.

And it is given that the breadth was decreased by 3 cms.

Thus now it is a question of ratios: two quantities in ratio 11 : 10 and the decrease is 3 cm.

Thus 11k – 10k = 3 i.e. k = 3 and the original and new breadth are 33 cm and 30 cm.

E.g. 12: When the prices of a commodity increased by 11.11%, I could purchase 4 mangoes less in Rs. 300. Find the original price of one mango.

One should realize that the expenditure before the price increase and after it is the same.

The ratio of original prices and new prices are 9 : 10

Thus the ratio of original quantity and quantity that can be bought now is 10 : 9

Since the actual decrease is given as 4 mangoes, the original number of mangoes that could be purchased is 40 in Rs. 300. This translates to a price of Rs. 7.5 per mango.

Exercise

27. The prices of wheat reduce by 11.11%. Find the amount of wheat that can now be purchased in the same amount that was sufficient to purchase 72 kgs of wheat at the earlier price.

    a. 81 kgs   b. 64 kgs   c. 100 kgs   d. 78 kgs

28. Because of a 25% decrease in the prices of mangoes, in Rs. 60 Ravi could purchase 4 more mangoes that what he could at original price. What is the original price of 1 mango?

    a. Rs. 3.25   b. Rs. 3.75   c. Rs. 4.25   d. Rs. 5

29. The length of rectangle increased by 20%. To maintain the area, the breadth had to be reduced by 3 cm. What was the original breadth of the rectangle?

    a. 12   b. 16   c. 14   d. 18

30. Traveling from home to office, if I increase my speed by 37.5% than my usual speed, I save 15 minutes. What is the time that I take at the higher speed?

    a. 55 mins   b. 40 mins   c. 35 mins   d. 30 mins

31. Since the prices of mangoes decreased by 8.33%, I could purchase 4 mangoes more in Rs. 132. What was the price of one mango before the decrease?

    a. Rs. 3.5   b. Rs. 3.25   c. Rs. 3   d. Rs. 2.75

Successive Percentage Changes

The population of a city was 80 lacs in year 2000. In the years 2001, 2002 and 2003 the population of the city increased by 20%, 16.66% and 10% respectively. What is the population of the city at the end of 2003.

In this question, there are three successive percentage increases. We say they are successive because for each percentage increase the base is the value after the previous increase. In the year 2001, population grew to 80 × 1.2 = 96 lacs. Now in 2002, the 16.66% increase is over this new value 96 lacs and not over 80 lacs. Thus population at the end of 2002 is $96 \times \frac{7}{6} = 102$ lacs and the population at end of 2003 is $102 \times \frac{11}{10} = 112.2$ lacs.

All the above can be combined into the single expression: $80 \times \frac{6}{5} \times \frac{7}{6} \times \frac{11}{10} = 112.2$

E.g. 13: The population of a city increased by 25%, 18.18% 10% in three consecutive census but decreased by 9.09% in the fourth census. If the population after the fourth census was 65 lacs, find the population before the first census.

Here one would realise the importance of multiplying factors. Since the population of the base year is not given, it would be cumbersome to work unless one can use the following:

$x \times \frac{5}{4} \times \frac{13}{11} \times \frac{11}{10} \times \frac{10}{11} = 65$ giving $x = 44$ lacs.

E.g. 14: If the population of a city changed by +12.5%, -12.5%, +6.66% over three consecutive years, find the total percentage change over the three years.

The net result of the three successive percentage changes is $\frac{9}{8} \times \frac{7}{8} \times \frac{16}{15} = \frac{21}{20}$ and $\frac{21}{20}$ is equivalent to a percentage increase of $\frac{1}{20}$ i.e. 5%.

Formula for two successive percentage change

For the equivalent percentage change of two percentages, we also have a formula. But if one has understood the use of multiplying factor thoroughly, one need not resort to the formula which becomes very cumbersome except for a few easy values.

The equivalent percentage change of two percentage changes a% and b% is $a + b + \frac{a \times b}{100}$% . The formula works well enough for percentage increase and also decrease if signs are taken correctly.

E.g. 15: What is the single percentage change equivalent to two percentage decreases of 10%.

Using multiplying factors, $\frac{9}{10} \times \frac{9}{10} = \frac{81}{100}$ i.e. decrease of $\frac{19}{100}$ i.e. decrease of 19%

Using the formula, the equivalent percentage change is $-10 - 10 + \frac{(-10) \times (-10)}{100} = -20 + 1 = -19\%$

.

One would go wrong if one assumes both the decreases as 10% and considering that positive sign represents a decrease. In that case the answer would be $10 + 10 + \frac{10 \times 10}{100} = 20 + 1 = 21\%$ and it would be wrong. So always work with a percentage decrease as negative and percentage increase as positive.

Practice

First practice finding the manipulating successive changes as fractions and then move on to worded problems

I. Find the net percentage change corresponding to successive percentage changes of:

i. +20% & +10%	ii. +20% & –10%	iii. –12% & –8%	iv. +10% & –10%
v. +20% & –20%	vi. –25% & +25%	vii. +12.5% & –11.11%	viii. –12.5% & +14.28%
ix. –16.66% & +10%	x. 20%, 25% & 16.66%	xi. –8.33% & –27.27%	xii. +12.5% & –12.5%

II. In each of the following find the value of *x*%,

i. The net effect of +20% and *x*% change is +40%

ii. The net effect of 20% and *x*% change is 10%

iii. The net effect of -10% and *x*% is +10%

iv. The net effect of +25% and *x*% is 0%

v. Prices increase by 16.66%. Consumption changes by *x*%. Expenditure increases by 40%

vi. Length decreases by 12.5%, breadth increases by *x*%. Area increases by 16.66%

vii. At a circus, ticket prices increased by 10%, number of persons visiting the circus fell by *x*%. Revenue decreased by 1%.

viii. Prices decreased by 20%, Quantity bought increased by *x*%. Expenditure remained constant.

Answers:

I.	i. 32%	ii. 8%	iii. –19.04%	iv. –1%	v. –4%	vi. –6.25%	vii. 0%
	viii. 0%	ix. –8.33%	x. 75%	xi. –33.33%	xii. –1.5625%		
II.	i. 16.66%	ii. –8.33%	iii. 22.22%	iv. –20%	v. 20%	vi. 33.33%	vii. –10%
	viii. 25%						

Exercise

32. If both the length and the breadth of a rectangle increases by 15%, by what percent does the area of the rectangle increase by?

    a. 30% b. 32.25% c. 20% d. 50%

33. Prices of a commodity decreased by 10% and hence the quantity sold by a trader increased by 8.33%. By what percent does the trader’s income increase or decrease?

    a. 2.5% decrease b. 4% decrease c. 2.5% increase d. 4% increase

34. Inspite of an increase in prices by 20%, Gopal managed to let his expenditures increase by just 10%. Find the percentage change in the quantity consumed.

    a. 9.09% b. 10% c. 8.33% d. 12.5%

35. Even though the prices of tickets increased by 20%, the revenue earned through sale of tickets increased by only 12%. Find the percentage change in the number of tickets sold.

    a. 8% b. 6.667% c. 10% d. 5%

Profit, Loss and Discount

Basic Terms of Profit/Loss

The topic Profit, Loss and Discount is a straightforward application of Percentages. There is very little new matter that one would study.

Selling Price

In most of the problems there would be a trader who sells goods to customer. The price at which the trader sells the goods is called the Selling Price (SP, in short).

Cost Price

The trader would either be purchasing the good from else-where or manufacturing it. In either of the cases, he would be incurring a cost. This is called his Cost Price (CP, in short)

Profit/Loss

To find if the trade has resulted in a profit or loss, we would always consider the difference (SP – CP).

If the SP is greater than the CP, as usually is the case, the difference is positive and we say that the trader has earned a Profit.

If the SP is less than the CP, the difference is negative and we say the trader has incurred a Loss.

Profit or Loss = (SP – CP), if positive, it's Profit, if negative it's Loss

Profit/Loss Percentage

Whenever we have to compare two products, the absolute value of the profit is not of much use. A product may be resulting in a huge profit, say of Rs. 1000, as compared to another product which results in a profit of only Rs. 100. While the first product may seem attractive because of the higher absolute value of profit, but if the first product costs Rs. 10,000 and the second product costs only Rs. 200, the scenario changes. Having spent Rs. 10,000 and making a profit of only 1,000 ($1/10^{th}$ of the investment) is not as lucrative as spending only Rs. 200 and making a profit of Rs. 100 (1/2 of the investment). Thus, to compare two products, a better measure would be $\frac{\text{Profit}}{\text{Cost}}$. This figure is usually expressed in percentage terms i.e. profit per Rs. 100 of cost. This is termed as Profit Percentage.

Thus, Profit Percentage = $\frac{\text{Profit}}{\text{Cost}} \times 100\ \%$

The same expression is used in the case of Loss Percentage as well. It's just that Profit in that case will be negative and the profit percentage will also be negative.

Profit is always expressed as a percentage of CP. Thus to find the Profit Percentage, the CP (and NOT the SP) will be the base.

> **E.g. 1:** Find the profit percentage if the CP is Rs. 450 and the SP is Rs. 500
>
> Obviously the profit = SP – CP = 500 – 450 = 50.
>
> Profit Percentage = $\frac{50}{450} \times 100$ i.e. $\frac{1}{9} \times 100$ i.e. 11.11%

E.g. 2: A trader purchases a good at Rs. 500 and is forced to sell it at Rs. 450. Find his profit percentage.

In this case there is a loss i.e. Profit = 450 – 500 = –50

Profit percentage = $\frac{-50}{500} \times 100$ i.e. $-\frac{1}{10} \times 100$ i.e. 10% loss

E.g. 3: Amit sells his watch for Rs. 200 for a profit of 20%. At what price did Amit purchase the watch?

Please note that 20% is of the CP which is not known and it is NOT of 200. Thus

CP + 20% of CP = 200

This is same as a 20% increase in CP results in 200. Thus using percentage increase fundas learnt in the earlier chapter, we have $\frac{6}{5} \times \text{CP} = 200 \Rightarrow \text{CP} = \frac{1000}{6} = 166.66$.

E.g. 4: Rohit sells his watch at a profit of 12.5% and in the bargain makes a profit of Rs. Rs. 15. Find the *CP* and *SP* of the watch.

Again 12.5% is of the *CP* and this amount is equal to Rs. 15. Since 12.5% in fraction terms is $\frac{1}{8}^{th}$, we have $\frac{1}{8}^{th}$ of CP = 15 i.e. CP = 120

Hence SP = 120 + 15 = 135.

E.g. 5: Ramesh sells his watch for Rs. 100 and makes a loss of Rs. 10. Find the loss percentage.

These easy questions are given just to rub in the fact that Profit or Loss percentage is always of CP and not SP. Thus in this questions 10% loss is wrong answer as 10 is found as a percentage of 100, which is the SP and not CP.

CP = 100 + 10 = 110

Thus Loss percentage = $\frac{10}{110} \times 100$ i.e. $\frac{1}{11}$ i.e. 9.0909%

Margin

Sometimes the profit is expressed as a percentage of SP. This percentage figure is called the Margin.

Thus, Margin = $\frac{\text{Profit}}{\text{SP}} \times 100$.

E.g. 6: If the profit percentage is 6.66%, find the margin percentage.

6.66% i.e. $\frac{1}{15}$ is of the CP, so it is best to assume the CP = 15. Hence profit = 1 and SP = 16.

And now margin percentage is $\frac{1}{16}$ i.e. 6.25%.

E.g. 7: If the margin percentage is –37.5%, find the profit percentage.

In this question, –37.5% is that of SP and NOT CP and further since it is negative it is a case of a loss. Since 37.5% is $\frac{3}{8}$, let us assume the SP to be 8 and hence loss = 3 and the CP = 11.

Hence the profit percentage is $-\frac{3}{11}$ i.e. –27.27%.

Exercise

1. If the cost price is 25% less than the selling price, find the profit percentage.

   a. 33.33% b. 25% c. 30% d. 20%

2. An article sold at Rs. 423 results in a loss of 10%. What should be the selling price to result in a profit of 10%?

   a. Rs. 446 b. Rs. 465 c. Rs. 517 d. Rs. 510

3. A man gains 10% when he sells the article at a certain price. Find his profit percentage if he increases the selling price by 50%.

   a. 65% b. 60% c. 55% d. 50%

4. Profit earned by selling an article at 1060 is 20% more than the loss incurred by selling the article for Rs. 950. Find the cost price of the article.

   a. Rs. 980 b. Rs. 1000 c. Rs. 1030 d. Rs. 1040

5. A shop-keeper sells an article at a loss of 5%. Had he sold it for Rs. 35 more, he would have made a profit of 15%. Find the cost price of the article.

   a. Rs. 175 b. Rs. 150 c. Rs. 190 d. Rs. 120

6. An article sold at a certain amount results in a loss of 7% whereas when it is sold at Rs. 100 more than earlier amount, it results in a 13% profit. Find the cost price of the article.

   a. Rs. 250 b. Rs. 380 c. Rs. 500 d. Rs. 150

Standard Problems on Profit and Loss

Chain of sale and purchase

E.g. 8: A manufacturer sells to the wholesaler at a profit of 20%. The wholesaler sells to the dealer at a profit of 12.5%. The dealer sells to a retailer at a profit of 10% and the retailer to the customer at a profit of 6.66%. If the cost of manufacturing the product for the manufacturer was Rs. 625, find the price paid by the customer.

All the profit percentages given are nothing by successive percentage increases on the cost of manufacturing. Thus the final price paid by the customer is $625 \times \frac{6}{5} \times \frac{9}{8} \times \frac{11}{10} \times \frac{16}{15} = 990$.

E.g. 9: A manufacturer sells to the wholesaler at a profit of 16.66%. The wholesaler sells to the dealer at a profit of 9.09%. The dealer sells to a retailer at a profit of 10% and the retailer to the customer at a profit of 14.28%. If the customer paid Rs. 468 for the article, find the cost of manufacturing the product incurred by the manufacturer.

In this question the final price is given and the base price, cost of manufacturing is not known. Thus if the cost of manufacturing is assumed to be *x*, we get

$$x \times \frac{7}{6} \times \frac{12}{11} \times \frac{13}{12} \times \frac{8}{7} = 468 \Rightarrow x \times \frac{13 \times 4}{3 \times 11} = 468 \text{ i.e. } x = 297.$$

Cost & Price given as Number of Articles/Rupee

E.g. 10: If the selling price of 10 articles is equal to the cost price of 9 articles, find the profit or loss percentage.

Method 1:

Let the SP of 10 articles = CP of 9 articles = Rs. k

SP of 1 article = Rs. $\frac{k}{10}$ and CP of 1 article = Rs. $\frac{k}{9}$

Using, $f = \frac{SP}{CP}$, we find the multiplying factor corresponding to the profit percentage as $\frac{9}{10}$ i.e. a loss of $\frac{1}{10}$ i.e. 10% loss.

Method 2:

To avoid all work involving fractions, a simple technique is to make use of LCM as follows:

Let the SP of 10 articles = CP of 9 articles = Rs. 10 × 9 say

SP of 1 article = Rs. 9 and CP of 1 article = Rs. 10

Thus profit percentage = $-\frac{1}{10}$ i.e. 10% loss.

E.g. 11: If the cost price of 16 articles is equal to the selling price of 15 articles, find the profit or loss percentage.

By the use of the LCM technique learnt above, CP of 1 article = Rs. 15 and SP of 1 article = Rs. 16. Thus profit percentage = 1/15 i.e. 6.66%

E.g. 12: By selling 12 articles, Naveen managed to make a profit equal to the selling price of 3 articles. Find the profit or loss percentage.

SP of 12 articles – CP of 12 articles = SP of 3 articles

SP of 9 articles = CP of 12 articles

Using the LCM approach, SP of 1 article = Rs. 12 and CP of 1 article = Rs. 9 and profit percentage is 3/9 i.e. 1/3 i.e. 33.33%

Alternately, $f = \frac{k/9}{k/12}$ i.e. $\frac{4}{3}$ i.e. a profit of 33.33%

E.g. 13: A person buys 12 oranges in a rupee. How many should he sell in a rupee so that he earns a profit of 20%.

CP of 1 orange = Rs. $\frac{1}{12}$

SP of 1 orange= Rs. $\frac{1}{12} \times \frac{6}{5} = \frac{1}{10}$

Thus 10 oranges have to be sold in 1 Re.

E.g. 14: A person buys 12 oranges for 5 Rs. How many oranges should he sell for Rs. 4 such that he makes a profit of 6.66%

CP of 1 orange = Rs. $\frac{5}{12}$

Profit of 6.66% will result in a multiplying factor of 16/15

Thus, SP of 1 orange = Rs. $\frac{5}{12} \times \frac{16}{15} = \frac{4}{9}$ and in Rs 4 he should sell $\frac{4}{4/9}$ = 9 oranges.

E.g. 15: A person buys oranges at the rate of 5 for Rs. 2 and sells them at a rate of 7 for Rs. 3. Find the profit or loss percentage.

Method 1: SP of 1 orange = Rs. $\frac{3}{7}$ and CP of 1 orange = Rs. $\frac{2}{5}$.

Using, $f = \frac{SP}{CP}$, we find the multiplying factor corresponding to the profit percentage as $\frac{15}{14}$ i.e. a profit of $\frac{1}{14}$ i.e. 7.14%.

Method 2: To avoid all fractions, one could also have made use of LCM as follows:

Let him buy and sell 35 oranges (LCM of 5 and 7). Thus, his cost will be Rs. 2 × 7 = Rs. 14 and earnings will be Rs. 3 × 5 = Rs. 15. Thus, his profit percentage is 1/14 i.e. 14.28%

Questions Involving Two or More Lots

E.g. 16: A trader sold one third of his stocks at a profit percentage of 10% and the rest at a profit percentage of 25%. Find his overall profit percentage.

Considering cost of his stock Rs. 300, he sold goods costing Rs. 100 at a profit of 10% and goods costing Rs. 200 at a profit of 25%. Thus his overall profit is Rs. 10 + Rs. 50 = Rs. 60.

Thus his overall profit percentage = 60/300 i.e. 1/5 i.e. 20%

Alternately, using funda of weighted averages (to be learnt in next chapter), profit percentage = $\frac{10\% \times 1 + 25\% \times 2}{1+2} = \frac{60}{3}$ = 20%

E.g. 17: Ankur purchased oranges in two lots, one at rate of Rs. 3 per orange and other at rate of Rs. 5 per orange. He sold them at a rate of Rs. 4 per orange. Find his profit or loss percent if ......

Case i: he purchases the same number of oranges in the two lots.

Case ii: he spends the same amount of rupees on the two lots.

Case i: Since he purchases the same number of oranges in both the lots, let him purchase 1 orange in each lot. Thus his cost = Rs. 3 + Rs. 5 = Rs. 8.

He now sells 2 oranges at the rate of Rs. 4 per orange. Thus he earns Rs. 8

Thus he neither makes a profit nor a loss.

Case ii: In this case, he is spending equal amounts on each lot. Thus let him spend Rs. 15 (the LCM of 3 and 5) on each lot. Thus he has incurred of cost of Rs. 30 and purchased 5 + 3 = 8 oranges.

He sells these 8 oranges at a rate of Rs. 4 per orange and earns Rs. 32.

Thus his profit percentage = 2/30 i.e. 1/15 i.e. 6.66%

E.g. 18: Ankur purchased oranges in two lots, one at rate of 3 oranges for a rupee and other at rate of 5 oranges for a Rupee. He sold them at a rate of 4 oranges per Rupee. Find his profit or loss percent if ……

Case i: he purchases the same number of oranges in the two lots.

Case ii: he spends the same amount of rupees on the two lots.

This question differs from the earlier example because here the rate is given as number of oranges per rupee and not rupees per orange.

Case i: Same number of oranges are purchased in each lot. First lot is of 3 oranges per rupee and second is of 5 oranges per rupee. Also the selling rate is of 4 oranges per rupee. Thus let him purchase the LCM of 3, 4 and 5 oranges in each lot i.e. 60 oranges in each lot. This would avoid all fractions.

So he purchases 120 oranges in all and incurs a cost of Rs. 20 + Rs. 12 = Rs. 32.

He sells these 120 oranges, 4 to a rupee and thus earns Rs. 30. Thus it is a case of loss.

Loss percentage = 2/32 i.e. 1/16 i.e. 6.25% loss.

Case ii: Let him spend 1 rupee on each lot. Thus he purchases 3 + 5 = 8 oranges after spending Rs. 2.

He sells the 8 oranges, 4 to a rupee and thus earns Rs. 2

So, it's a case of him not earning any profit nor losing any money.

E.g. 19: Rohan buys two commodities for Rs.60000 each. He sells one at a profit of 20 % and sells the other at a loss of 20 %. Find his overall profit or loss percent and also the amount of profit or loss.

Since the CP of both the articles are the same, +20% of 60,000 and –20% of 60,000 will cancel out each other and he will neither make any profit nor any loss.

E.g. 20: Rohan sells two commodities for Rs. 19,8000 each. He sells one at a profit of 10 % and sells the other at a loss of 10 %. Find his overall profit or loss percent and also the amount of profit or loss.

In case like this when the SP of two articles are the same and one is sold at a profit of x% and other at a loss of x%, there would always be a loss of $\frac{x^2}{100}$ %

Thus loss percent in this case = $\frac{10^2}{100}$ = 1% loss.

To calculate the amount of loss, we would have to find the CP of the articles. Thus,

$$CP_1 \times \frac{11}{10} = 19{,}800 \Rightarrow CP_1 = 18{,}000 \text{ and } CP_2 \times \frac{9}{10} = 19{,}800 \Rightarrow CP_2 = 22{,}000$$

Thus total cost = Rs. 18,000 + Rs. 22,000 = Rs. 40,000

Total earnings = Rs. 19,800 + Rs. 19,800 = Rs 39,600.

Thus loss = Rs. 400.

Alternately after finding the costs, loss = 1% of total cost = 1% of 40,000 = Rs. 4,000

E.g. 21: Rohan sells two commodities for Rs. 12,000 each. He sells one for a profit of 20% and other at a loss of 11.11%. Find his overall profit or loss percentage and also the amount of profit or loss.

While the selling price of the two commodities in this question is the same, but the profit % and loss % on the two articles is not the same numeric percentage and hence we cannot use the short-cut learnt in earlier question. In this case, we would have to work out the costs and only then find the profit/loss percentage.

$CP_1 \times \frac{6}{5} = 12,000 \Rightarrow CP_1 = 10,000$ and $CP_2 \times \frac{8}{9} = 12,000 \Rightarrow CP_2 = 13,500$

Thus total cost = Rs. 23,500 and total earnings = Rs. 24,000.

Thus profit percentage = 500/23,500 i.e. 1/47 i.e. $\frac{1}{47} \times 100$ i.e. approximately 2.1%

With Changes in the CP and SP

E.g. 22: When the cost price of an article increases by 20%, a shop-keeper also increases his selling price by 20%. Find the change in the profit percentage and also in the amount of profit.

The profit percentage is solely dependent on the ratio $\frac{SP}{CP}$. With a 20% increase in both CP and SP, the ratio will not change and hence the profit percentage will also not change.

Hence we see that if both cost price and selling price are increased by the same percentage, the profit percentage remains the same. Since profit percentage is the same, but cost price has increased, so the amount of profit would also have increased. Let's see by how much.

Earlier Profit = SP – CP

New Profit = 1.2SP – 1.2CP = 1.2 × (SP – CP) i.e. 1.2 times the earlier profit.

Exercise

7. On selling 20 mts of cloth, a merchant realizes that he has made a loss equal to the selling price of 4 mts of cloth. Find his loss in percentage terms.

   a. 20% b. 16.66% c. 15% d. 12.5%

8. If the cost price of 30 articles is equal to the selling price of x articles, find the value of x that will result in a profit of 20%

   a. 25 b. 35 c. 24 d. 36

9. A trader sells 10 articles for a rupee and manages a profit of 20%. How many articles did he purchase in a Rupee?

   a. 11 b. 13 c. 10 d. 12

10. A trader purchases orange at the rate of 1 dozen for Rs. 5. How many oranges should he sell per Rupee such that he makes a profit of 20%?

    a. 1/2 b. 2 c. 1/4 d. 4

11. A trader buys 20 kgs of wheat at the rate of Rs. 6 per kg. He sells the wheat at the rate of 8 per kg. But, for a regular customer his rate is Rs. 5 per kg. On selling the entire 20 kgs, he realizes he has not made any profit or loss. How many kgs were bought by regular customers?

    a. 13.33 kgs b. 10 kgs c. 15 kgs d. 12 kgs

12. A trader buys articles at the rate of 10 per rupee. He sells one third of the lot at the rate of 12 per rupee and the rest at the rate of 9 per rupee. Find his profit percentage.

    a. 10% b. 15% c. 0% d. 5%

13. Articles were bought at the rate of 6 for Rs. 5 and sold at the rate of 5 for Rs. 6. Find the profit percentage.

    a. 40% b. 20% c. 15% d. 44%

14. A shop-keeper sells two articles, each for Rs. 1980. If he sold one at a profit of 10% and the other at a loss of 10%, find the amount of profit or loss.

    a. Rs. 40 loss b. Rs. 20 gain c. Rs. 10 gain d. Rs. 0

Mark-Up and Discount

Since customers haggle for a discount, the usual practice is that shop-keeper's prepare the price tag that is higher than the acceptable price at which the shop-keeper is willing to sell at. Then a discount is offered on the Marked Price (MP), also called List Price and the article is sold at a Selling Price lower than the Marked Price. The profit is still calculated based on the final Selling Price and the Cost Price. The associated terms are defined below:

Marked Price:

This is the price that is marked on the price tag

Markup:

The percentage by which the MP is higher than the CP. Thus mark-up percentage is a percentage of the Cost Price.

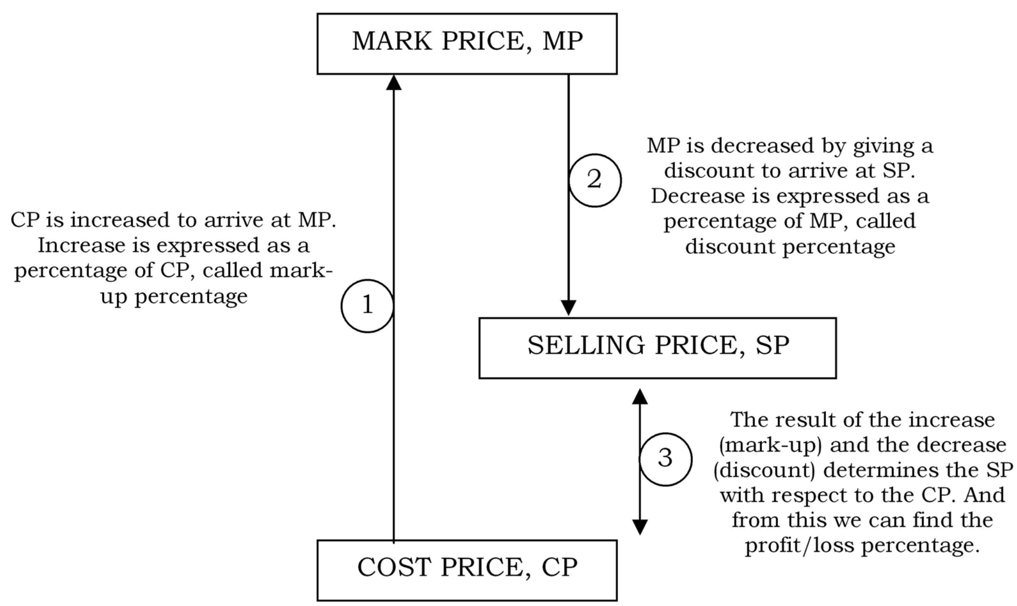
Mark-up should be viewed as just a percentage increase is. Thus, for a particular mark-up percentage, there would be an associated multiplying factor, say fm and we would have the relation CP × f_m = MP

Discount:

Discount is the amount by which the MP is lowered while selling due to sales promotion or due to bargaining. It is usually expressed as a percentage and is a percentage of the Marked Price.

Discount should be viewed as just a percentage decrease is. Thus, for a particular discount percentage, there would be an associated multiplying factor, say fd and we would have the relation MP ×f_d = SP

The diagram pictorially represents the above terms:



Mark-up & Discount as case of Successive Percent Changes

Let's denote mark-up percentage by $m\%$ and discount percentage by $d\%$.

It should be apparent that

$m\%$ is a percentage increase over the CP to result in the MP

d % is a percentage decrease over MP to result in SP.

Thus, first CP is increased by m% to result in the MP and then the MP is decreased by d % to result in the SP i.e. m % and d % are two successive percentage changes acting on CP to result in SP

Thus using the formula learnt for successive percentage changes in the chapter on percentages we can say that:

Profit percentage = $m - d - \frac{m \times d}{100}$ %

As stated in chapter of percentages, the above is useful only when mark-up and discount are good numbers like 10% or 20%. If they are not so good numbers like 12.5% or 16.66%, using multiplying factor is strongly recommended.

E.g. 23: A trader marks-up his goods by 30% and then offers a discount of 10%. Find the net profit or loss percentage that he makes.

Profit percentage = $30 - 10 - \frac{30 \times 10}{100}$ = 17% we have

E.g. 24: A trader marks-up his goods by 27.27% and then offers a discount of 8.33%. Find his profit percentage.

Since the numbers are not very amenable in the formula, working on multiplying factors,

$f_p = \frac{14}{11} \times \frac{11}{12} = \frac{7}{6}$. Thus, the profit percentage is 1/6 i.e. 16.66%

The above should be self-sufficient, but if you are not convinced about $f_p = f_m \times f_d$, see if the following helps.

$$\underbrace{\left(CP \times \underbrace{\frac{14}{11}}_{\substack{\text{marking-up} \\ \text{\% inc of 3/11}}}\right)}_{\text{Mark Price}} \times \underbrace{\frac{11}{12}}_{\substack{\text{discounting} \\ \text{\% dec of 1/12}}} = SP$$

i.e. $CP \times \frac{7}{6} = SP$ which suggests a profit percentage of 1/6 i.e. 16.66%

E.g. 25: Inspite of giving a discount of 9.09%, a trader made a profit of 11.11%. By what percentage did he mark-up his goods?

Using $f_p = f_m \times f_d$, we have $\underbrace{\frac{10}{9}}_{\text{profit 1/9}} = f_m \times \underbrace{\frac{10}{11}}_{\text{discount 1/11}} \Rightarrow f_m = \frac{11}{9}$. A multiplying factor of 11/9 means a percentage increase of 2/9 i.e. 22.22%

Exercise

15. A 25% discount offer results into a saving of Rs. 37. Find the selling price of the article.

a. 111 b. 148 c. 160 d. 185

16. A trader gives two successive discounts of 20% and 10%. What is the equivalent discount that he is offering?

a. 30% b. 15% c. 32% d. 28%

17. A scheme of 1 soap free with every 4 soaps purchased is launched for increasing the sales. What is the effective discount that the scheme offers?

a. 20% b. 25% c. 30% d. 33.33%

18. As a sales incentive, which of the following two schemes should a shampoo manufacturer prefer over the other?

I: Offer to give 25% more quantity for the same price;

II: A discount of 25% on the price.

a. I b. II c. Either both are same

19. What should be the mark-up percentage if a trader wishes to make a profit of 10% inspite of a discount of 20%

a. 10% b. 30% c. 20% d. 37.5%

20. Find the ratio of the marked prices of two articles whose selling prices are same after they are sold at a discount of 12.5% and 9.09% respectively.

a. 80 : 77 b. 64 : 63 c. 72 : 70 d. 11 : 8

Simple and Compound Interest

Rate of Interest, Principal and Time

When we part with our money to keep it in the bank, the bank pays us an ‘interest’ to do so. Similarly when we borrow money, we have to pay an ‘interest’. The interest received or paid depends on three factors:

Principal (P): The amount of money that is loaned out or is borrowed is called the Principal. Interest would obviously depend on the principal. Higher is the amount, more is the interest and lower the amount, lesser is the interest.

Time Period (t): Again it should be obvious that if the money is loaned or borrowed for a longer time period, more interest will be charged and for a shorter time period, lesser will be the interest.

Rate of Interest (r): There is a third factor which determines the Interest. This is the Rate of Interest, expressed as a percentage. This is a pre-determined amount that the giver and the receiver agree to and it specifies the amount of interest charged per Rs. 100 of the principal for a specified time period. Thus if the rate of interest agreed is 8% per annum, it means that on every Rs. 100 of the principal amount, the interest will Rs. 8 per year.

Further, there are two ways in which an Interest can be charged. For each of the year the money is borrowed/loaned, Interest could be charged just on the principal amount loaned/borrowed (as done in Simple Interest) or for each of the year the interest could be charged on the amount outstanding i.e. principal amount and the accrued interest of the previous years (as done in Compound Interest). These two methods are discussed in details in the following sections.

Simple Interest

In this method the interest charged per year is calculated only on the principal. What this means is that if Rs. 1000 will be loaned/borrowed, the interest will be charged only on Rs. 1000 even in successive years.

Thus if the rate is r% per annum, then Simple Interest per year = r% of P and thus we get the following formula for *SI*, which most of us are already acquainted with.

$$\text{SI} = \frac{\text{P} \times r \times t}{100}$$

Amount returnable = Principal + SI

E.g. 1: If A deposits Rs. 20,000 in a bank for 3 years at a rate of 10%, what is the simple interest he will get at the end of the period?

Here P = 20,000, r = 10%, t = 3 yrs

Hence SI = $\frac{20{,}000 \times 10 \times 3}{100}$ = Rs. 6,000

E.g. 2: Sagar borrows Rs. 50,000 from a bank for 5 years. What is the rate of simple interest charged by the bank if after 5 years Sagar had to pay Rs. 66,000 to the bank?

Here 66,000 is the total outstanding amount which is Principal + SI. Since principal is given as Rs. 50,000, we get SI = 66,000 – 50,000 = Rs. 16,000

Using the above formula, $16{,}000 = \frac{50{,}000 \times r \times 5}{100} \Rightarrow r = \frac{32}{5}$ = 6.4%

E.g. 3: After how many years would an amount double itself at 15% rate of interest?

We don't have the principal here, we can denote it by P. If the amount doubles then obviously the SI would be equal to the principal.

So we have $P = \frac{P \times 15 \times t}{100} \Rightarrow t = \frac{100}{15}$ = 6.667 years

E.g. 4: The rate of interest of a bank is 10%. Akash was taking a loan of a huge amount for one year so the bank agreed to give him the loan at a rate of 8%. They reasoned that even with 8% they will get twice the interest that they would have got had they given out a loan of Rs. 40,00,000 at 10% for a year. What is the amount of Akash's loan?

If P is the loan taken by Akash, writing the realtion comparing the two simple interests,

$$\frac{P \times 8 \times 1}{100} = 2 \times \frac{40,00,000 \times 10 \times 1}{100} \Rightarrow P = 1,00,00,000$$

Exercise

1. Ajay takes a loan of Rs. 30,000 from a bank for 8 years at 6.5% rate of simple interest. He then loans out Rs. 20,000 for 8 years at 7.5% rate of simple interest. He could loan out the balance only at 5.5% for 8 years. In the entire transaction, did Ajay make or lose money and how much?

   a. Rs. 400 gain b. Rs. 800 gain c. No gain, No loss d. Rs. 400 loss

2. Vijay took part of Rs. 10,000 loan at 4% and the rest at 6%. If he pays a total interest of Rs. 900 in two years, find the amount taken on loan at 4%. The interest rate charged by the bank is Simple Interest.

   a. Rs 8000 b. Rs 7500 c. Rs 7000 d. Rs 6000

3. The underworld don Chhota Pappu loans money to people at simple interest. He charges a certain rate of interest for the first year. Next year he doubles the initial rate of interest on the amount. Third year he triples the initial rate of interest and so on... A man took an amount of Rs. 9000 and after 3 yrs paid back an amount of Rs. 15000 back. What was the rate of interest in the first year?

   a. 9% b. 9.09% c. 10% d. 11.11%

4. I needed Rs. 1,20,000 to buy a Plasma TV and hence I borrowed Rs. 75,000 from Vani and the rest from Vivek. Vani and Vivek charge me a rate of interest such that the interest amount payable to both of them is the same. If in all I re-pay them a total of Rs. 1,50,000 at the end of 2 years, what is the rate of interest charged per annum by the two?

   a. 10%,20% b. 12.5%, 20% c. 10%,16.66% d. 12.5%, 16.66%

5. At a certain rate of simple interest, a principal becomes three times in 15 years. In how many years will the principal amount become nine times?

   a. 45 years b. 30 years c. 60 years d. 75 years

Compound Interest

In the case of Compound Interest, after a fixed time intervals (again pre-determined and agreed to by the giver and receiver) the interest amount is calculated on the principal and this interest amount is added to the principal to determine the current outstanding amount. Then during the next time interval, Interest is charged on the current outstanding amount and not on the principal. Again this interest is added to find the outstanding amount. This process is continued for successive time intervals.

Explanation of the above using a numeric example:

Consider a loan of Rs. 1000 is taken with compound interest being charged at a rate of 10% p.a. Read the following table row-wise.

Year	Amount outstanding at start of year	Interest charged this year	Amount outstanding at end of year
1st year	1,000	100	1,100
2nd year	1,100	110	1,210
3rd year	1210	121	1,331
4th year	1,331	133.1	1,464.1

The formula in case of CI is: $A = P\left(1+\frac{r}{100}\right)^n$, where r is the agreed rate for a predefined interval of time and n is the number of time periods the loan is taken or amount is deposited for.

This formula gives us the final outstanding amount and NOT the Compound Interest.

Compound Interest = Amount – Principal

E.g. 5: A bank charges a rate of interest of 10% compounded annually. What is the total amount to be paid on a loan of Rs. 36000 for 2 yrs?

Using the above given formula, $A = 36{,}000\left(1+\frac{10}{100}\right)^2 = 36{,}000 \times \frac{121}{100}$ = Rs. 43,560

E.g. 6: A man takes a loan of Rs. 1,00,000 for two years at compound interest. If he has to return Rs. 1,10,250, find the rate of interest charged.

Using the formula we have 1,10,250 = 1,00,000 $\left(1+\frac{r}{100}\right)^2$

Doing this calculation and finding the square root is going to be a cumbersome process. The better alternative is to make an educated guess and then confirm this by squaring as follows:

The man pays an interest of Rs. 10,250 on a principal of 1,00,000. Thus the interest paid in two years is a little over 10%. A good guess would be that the interest rate per annum is 5%. Assuming it is 5% and finding the square of 1.05 we see that $1.05^2 = 1.1025$.

Thus we can confirm that the interest rate was indeed 5%.

E.g. 7: Manu lends a sum of money to his friend at interest such that the amount triples after 5 years when compounded annually. In how many years would the amount payable back become 9 times the sum loaned?

Here Amount becomes thrice the Principal. So, $3P = P\left(1+\frac{r}{100}\right)^5 \Rightarrow \left(1+\frac{r}{100}\right)^5 = 3$

The question requires us to find n when $9P = P\left(1+\frac{r}{100}\right)^n$ i.e. when $\left(1+\frac{r}{100}\right)^n = 9$.

Denoting $\left(1+\frac{r}{100}\right)$ as f, we have $f^5 = 3$ and we want to find $f^n = 9$. Obviously since 9 is square of 3, we will have $f^{10} = 9$. Thus the amount will become 9 times the principal in 10 years.

Alternate SHORT method: In the case of compound interest, it is a fresh beginning after every compounding. Thus after 5 years, when the principal triples, it is as good as a fresh beginning. In the next 5 years this amount, 3P, will again triple i.e. will become 9P. Thus, in a total of 10 years, the amount will become 9 times.

E.g. 8: A man loans out Rs. 50,000 at a rate of 8% simple interest for 2 years and another Rs. 50,000 at the same rate compounded annually for 2 years. Is there any difference in the two amounts he gets back after 2 years?

Lets first calculate the total amount in the case of Simple Interest

$SI = \frac{50,000 \times 8 \times 2}{100}$ = Rs. 8000

Now, let's calculate the total amount in the case of Compound Interest

$A = 50,000 \times 1.08^2 = 50,000 \times 1.1664$ = Rs. 58,320

This is the amount and not the CI. The CI = 58,320 – 50,000 = Rs. 8,320

Hence there is a difference of Rs. 320 in the two amounts he gets back

Non-Annual Compounding

Compounding is the process of adding the interest accrued to the principal. In the case Rs. 1000 was kept at 10% compound interest, the interest was added back to the principal at the end of every year. Hence it is called annual compounding.

However note that the interest of Rs. 100 earned in the 1st year is not earned on the 365th day of the year. It is continuously being earned throughout the year. Thus, after 6 months, the interest earned would have been Rs. 50. However one waits for the entire year to pass before adding it back to the principal, because it is a case of annual compounding.

Had the giver and receiver agreed on a 6-months compounding, called semi-annual compounding, the time period would have been measured in spans of 6 months and the principal would be increased by the interest accrued every six months.

Compounding can be any time period – annually, semi-annually, quarterly, monthly, daily.

The formula for non-annual compounding remains the same, $A = P\left(1+\frac{r}{100}\right)^n$. The only difference being that here r will be the rate of interest per 'compounding period'. If in a question, it is given that compounding happens half yearly but r is given as the rate per annum, then we will need to divide r by 2 to get half yearly rate of interest. Similarly for quarterly compounding, if r is given as the rate per annum, then we will need to divide r by 4 to get quarterly rate of interest.

Also remember that t here is the number of 'compounding periods'. So if the interest is compounded half yearly, and number of years, n is given, the number of time periods will be equal to $2 \times n$.

E.g. 9: If interest is accrued semi annually at 10% per annum, what will be the total amount at the end of 2 years if principal is Rs. 15000.

Since the interest is compounded semi annually and rate of interest is per annum, the semi annual rate of interest will be 10/2 = 5%. Also t will be 2 × 2 = 4.

So $A = 15{,}000\left(1+\frac{5}{100}\right)^4 = 15{,}000 \times \left(\frac{21}{20}\right)^4$ = Rs. 18,233

E.g. 10: If a bank offers two schemes (i) Annual compounding at 11% (ii) Semi-annual compounding at 10%, which of the two is a better scheme for depositors who want to deposit their money for two years?

Let's assume principal to 100.

Case (i) In case of annual compounding r = 11%, t = 2

So $A = 100 \times 1.1^2 = 121$.

Case (ii) In case of Semi-annual compounding r = 10/2 = 5%, t = 4

So $A = 100 \times 1.05^4 = 100 \times 1.1025^2$.

One should not calculate and find the square. Remember we just need to compare and it is obvious that 1.1025^2 will surely be greater than 1.1^2.

Hence Case (ii) of semi-annual compounding is better for the depositors.

It should be obvious that people seeking loans would prefer Case (i) since they will need to pay less interest at the end of the year.

Note: Population, Appreciation and Depreciation are generally calculated at compound rate of interest unless otherwise stated. In most of the other cases unless otherwise stated, we will assume simple rate of interest.

Unless otherwise stated, compound interest will be compounded annually.

Exercise

6. What will the amount be after 3 years if I deposit Rs. 5000 in a bank which offers me a rate of interest of 5%.

   a. Rs. 5750 b. Rs. 5760 c. Rs. 5770 d. Rs. 5790

7. The population of a city grows at a rate of 5% per annum. If in 2006 its population is 1852200, what was its population in 2004?

   a. 12,60,000 b. 15,60,000 c. 16,00,000 d. 16,80,000

8. The property prices appreciate at a rate of 7% per annum. I bought a house in the year 2003 which had cost me Rs. 10,00,000 at that time. What will be its cost three years later?

   a. Rs. 12,23,000 b. Rs. 12,25,000 c. Rs. 12,27,000 d. Rs. 12,29,000

9. I bought an Astra two years back. Its value depreciated by 9% every year. If at present its value is Rs. 9,10,910, at what cost had I bought it?

   a. Rs. 10,91,910 b. Rs. 10,19,190 c. Rs. 10,00,000 d. Rs. 11,00,000

10. On investing Rs. 5000 in a bank, you will get back Rs. 5671 in 2 years. What is the compound rate of interest?

    a. 6.5% b. 6% c. 5.5% d. 5%

Compound Interest as a Case of Interest on Interest

In competitive exams there are many questions involving comparing simple interest and compound interest in the first two years or comparing the compound interest in two successive years. So let's look at this more thoroughly:

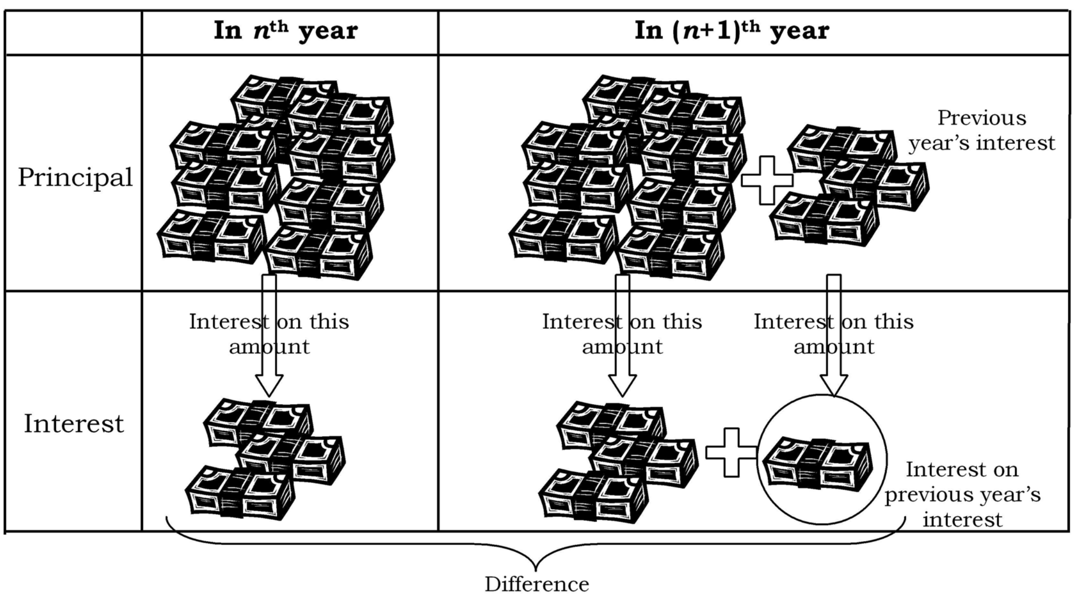
Compound Interest In Successive Years

In the case of compound interest the amount on which interest is calculated, keeps changing. The first year, it is equal to the principal, next year it is equal to principal plus last year's interest, in the year after that, interest is calculated on principal plus the interest earned in the previous two years and so on. This is what we mean by compound interest. The interest gets compounded (added to the principal) every year. Every subsequent year, the interest calculated for that year, will be more than the interest calculated for the previous year because the amount on which interest is calculated would be more than the amount of the previous year.

The following figure compares the interest earned in two successive years, n^{th} and $(n + 1)^{th}$. The picture is self sufficient to understand that, in any year, one would receive as much interest as earned in the previous year PLUS one would earn interest on the previous year's interest.

And this should be logical because the previous year's interest gets added to the amount at end of previous year.

Thus, **difference in compound interest earned in two successive years is equal to the interest on the first year's interest**.



Formulaic Approach: **the ratio of compound interest earned in two successive years is (1 + *r* %) : 1**

E.g. 11: If the ratio of CI in the 7^{th} and 8^{th} year is 10 : 11, find the rate of interest being offered.

Rather than use the formula for the ratio, the CI in the two years can be assumed as $10k$ and $11k$. Needless to say the extra k interest earned is on $10k$ and hence the rate of interest is 1/10 i.e. 10%

Using the formula, $\frac{11}{10} = \frac{1+r}{1} \Rightarrow r = \frac{11}{10} - 1 = \frac{1}{10}$ i.e. 10%

Exercise

11. The CI earned in the 7th year is Rs. 500. If the rate of interest is 15%, find the compound interest earned in the 8th year.

    a. Rs. 500 b. Rs. 525 c. Rs. 550 d. Rs. 575

12. The compound interest earned in the 3rd and 4th year is Rs. 450 and Rs. 500. Find the rate of interest.

    a. 9.09% b. 10% c. 11.11% d. 12.5%

13. At a compound interest rate of 10%, the compound interest earned in the 8th year is Rs. 484. Find the compound interest earned in the 6th year.

    a. Rs. 360 b. Rs. 400 c. Rs. 440 d. Rs. 480

14. If the rate of compound interest is 12.5%, find the ratio of compound interest earned in the 24th year and that earned in the 25th year.

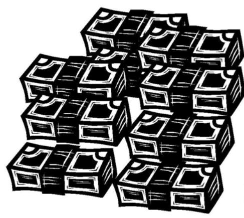
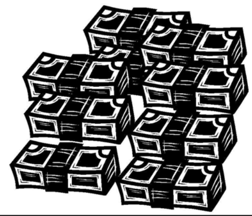
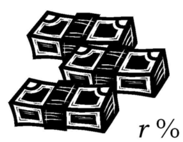
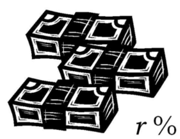
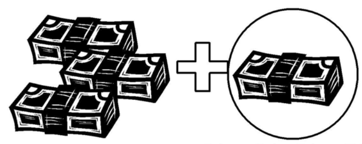
    a. 8 : 9 b. 9 : 8 c. 24 : 25 d. 25 : 24

15. If the ratio of compound interest earned in the n^{th} and the $(n+1)^{th}$ year is 15 : 16, find the rate of interest.

    a. 12.5% b. 13.33% c. 6.66% d. 6.25%

Difference Between SI And CI In First Two Years

Consider a principal kept at simple interest at certain rate. Also consider the same amount kept at same rate but this time at compound interest (annually). Now let us compare the two cases over the first two years.

	Simple Interest	**Compound Interest**
Principal	 P	 P
Interest 1st Year	 r % of P	 r % of P
Interest 2nd Year	 r % of P	 r % of P + r % of (r % of P)

Difference between total CI and total SI earned in first two years = Interest on the first years interest i.e. r % of I_1. But I_1 = r % of P. Thus, difference between the total CI and SI earned in first two years = r % of (r % of P) i.e. $\frac{r^2}{100}$ % of P

Ratio of CI and SI earned in first two years:

$$\frac{\text{CI earned in first 2 years}}{\text{SI earned in firsts 2 years}} = \frac{2\times(r\%\text{ of P})+r\%\text{ of }(r\%\text{ of P})}{2\times(r\%\text{ of P})} = \frac{2+r\%}{2}$$

E.g. 12: What is the difference between SI and CI accrued in the 2nd year (only in 2nd year, not 1st and 2nd combined) on a principal of Rs. 1000 at 8% interest?

Since the interest earned in 1st year is same whether it is SI or CI, the difference between the total CI and SI earned in first two years is actually the difference between the CI and SI of the 2nd year itself. Thus required difference = $\frac{8^2}{100\times100}\times1000$ = Rs. 6.40

E.g. 13: If the rate of interest is 15%, what is the ratio of the total CI earned in first 2 years to the total SI earned in first two years if the principal kept is same?

It's a straightforward application of formula found earlier i.e. $\frac{2+0.15}{2}=\frac{2.15}{2}$ i.e. 43 : 40

E.g. 14: If the ratio of the total compound interest earned in firsts two years to the total simple interest earned in first two years is 11 : 10, find the rate of interest, assuming it and the principal to be the same in the two cases.

Rather than using the formula, one could also think on following lines:

Let total SI earned in 2 years be $10k$. Thus, in the first year the SI earned, as well as the CI earned will be $5k$. Thus, in the second year, the CI earned will be $11k - 5k = 6k$.

In the second year, the CI earned is more than the first years CI of $5k$ by k. Thus, rate of interest is $k/5k$ i.e. 1/5 i.e. 20%

Exercise

16. The difference between the compound interest and simple interest on a certain sum at 10% per annum for 2 years is Rs. 631. Find the sum.

    a. Rs. 6,310 b. Rs. 63,100 c. Rs. 6,31,000 d. Rs. 63,10,000

17. The difference between the compound interest and simple interest accrued on an amount of Rs. 18,000 in 2 years was Rs. 405. Find the rate of interest, if it is same in the case of simple and compound interest.

    a. 10% b. 12.5% c. 15% d. 17.5%

18. I kept Rs. 20,000 at 5% rate of simple interest for two years. Find the difference in interest earned if I had kept the same amount for same years at same rate but at compound interest.

    a. Rs. 2 b. Rs. 5 c. Rs. 20 d. Rs. 50

19. If the rate of interest in case of both compound and simple interest is 8.33%, find the ratio of the compound interest and simple interest earned in first 2 years on the same principal.

    a. 25 : 24 b. 24 : 25 c. 11 : 12 d. 12 : 11

20. If the ratio of compound interest and simple interest earned in the first two years at the same rate on the same principal is 11 : 10, find the rate of interest.

    a. 10% b. 9.09% c. 20% d. 18.18%

Assignment: Percentages, PLD, SI CI

1. 6% of 18 is equal to 9% of ____?

   (a) 9 (b) 12 (c) 30 (d) 24

2. 22.22% of N is equal to 36.36% of 1210. Find N?

   (a) 900 (b) 980 (c) 990 (d) 1000

3. 18% of Cosco balls out of 120 balls are defective and 6% of Vicky balls out of 60 balls are defective. In all what % of total balls are defective?

   (a) 12% (b) 24% (c) 14% (d) 8%

4. A is 37.5% more than B. B is what percentage less than A?

   (a) 33.33% (b) 32.5% (c) 30% (d) 27.27%

5. Ram's income is 25% more than Lakshman's and Lashman's income is 20% more than Bharat's. By what percentage is Bharat's income less then Ram's?

   (a) 50 (b) 33.33 (c) 25 (d) 20

6. Instead of increasing a number by 25%, a student decreased the number by 25%. By what percent is his answer less than the correct answer?

   (a) 50% (b) 66.66% (c) 40% (d) 25%

7. The mileage (km/lt) of my car after servicing increased by 10%. By what percentage does my expenditure on petrol reduce because of this increase in mileage?

   (a) 9% (b) 9.09% (c)10% (d) 11.11%

8. A carpenter increases the length of a rectangular photo-frame by 8% and reduces the width by 10%. Overall percentage change in the area of the frame is:

   (a) –2% (b) 2% (c) 18% (d) –2.8%

9. If a number N is increased by 20% and the increased value is further increased by 20%, the total increase in the number is equal to 1760. What is the value of the number N?

   (a) 3860 (b) 3760 (c) 3960 (d) 3230

10. King Dashrath donated 20% of villages to pundits on receiving which pundits blessed him by saying "you shall receive same % of the number of villages left with you now". If Dashrath had 125 villages in his kingdom earlier, how many does he have now after the pundits blessing came true?

    (a) 130 (b) 120 (c) 125 (d) 100

11. Population of a town named Haripur increased by 10% in 2009 and increased by 20% in 2010 and is expected to reduce by 10% in 2011. What is the overall increase in the population of 2011 as compared to the population in 2008?

    (a) 18.8 (b) 20 (c) 15 (d) 10

12. Prices of chocolates increased by 10%. Because of this, I am now able to purchase 4 chocolates less in Rs. 100. Find the increased price of each chocolate?

    (a) Rs. 1 (b) Rs. 1.50 (c) Rs. 2 (d) Rs. 2.5

13. What will be the difference between amounts received in the following two schemes:

    Scheme I: Rs.1200 kept at SI for two years at 20% p.a.

    Scheme II: Rs.1200 kept at CI for two years at 20% p.a.

    (a) Rs. 48 (b) Rs. 24 (c) Rs. 0 (d) Rs. 12

14. A certain sum when kept at SI amounts to 7 times itself in 3 years. In how many years will it amount to 17 times?

    (a) 10 (b) 8 (c) 12 (d) 15

15. To double my investment in two years with interest being compounded annually, at what approximate rate of interest (p.a.) should I invest at?

    (a) 42% (b) 21% (c) 50% (d) 36%

16. A certain sum amounts to Rs. 144 in 4 years and the same sum amounts to Rs.100 in 2 years when compounded annually. What is the rate of interest being offered in the scheme?

    (a) 10 (b) 20 (c) 30 (d) 15

17. If a compound interest scheme doubles the sum in 3 years, in how many years will it give a return of 700%?

    (a) 6 years (b) 9 years (c) 12 years (d) 21 years

18. The interest on a certain sum when kept at SI for 2 years is Rs. 160 and when the same sum is kept at the same rate of interest, but compounded annually, the interest earned in 2 years is Rs. 190. What is the rate of interest?

    (a) 18% p.a. (b) 37.5% p.a. (c) 20% p.a. (d) 25% p.a.

19. What is the approximate sum kept in the previous question?

    (a) Rs. 213 (b) Rs. 320 (c) Rs. 2130 (d) Rs. 3200

20. I kept my money at compound interest of 20%, compounded annually and received an interest of Rs. 1000. Had the interest rate been 20% compounded semi-annually, by what amount would the interest earned have been higher?

    (a) 110 (b) 100 (c) 210 (d) 120

21. Aman buys grapes at Rs. 12.50/100 grams and sells it at Rs. 130/Kg. What is his profit percentage?

    (a) 8% (b) 4% (c) 10% (d) 12%

22. If the selling price of an article is $11/8^{th}$ its cost price, what is the profit percentage?

    (a) 27.27% (b) 30% (c) 45.45% (d) 37.5%

23. *A* sells goods to *B* at a profit of 20% and *B* sells same goods to *C* at a discount of 10%. If in the whole transaction the difference between cost price of *A* and *C* is Rs. 24, what is the cost price of *B*?

    (a) Rs. 300 (b) Rs. 360 (c) Rs. 124 (d) Rs. 240

24. If the selling price of 6 bananas is equal to cost price of 8 bananas then what is the profit/loss percentage?

    (a) 25% (b) 33.33% (c) 50% (d) 30%

25. If on selling eight articles, the loss incurred is equal to selling price of 3 articles, what is the loss percentage?

    (a) 37.5% (b) 62.5% (c) 40% (d) 27.27%

26. On selling an orange at 25 paisa, the vendor looses 16.66%. To gain the same percentage, the orange must be sold at how many paisa?

    (a) 30 (b) 42 (c) 35 (d) 40

27. An article A is priced at 8 for Rs. 12 and article B is priced at 12 for Rs. 8. A person buys equal quantity of both the articles and then makes packets of a pair of A and B. If he sells these packets at a rate of 6 packets for Rs. 13, then what is his approximate profit percentage?

    (a) 10% (b) 9.09% (c) 8.33% (d) 0%

28. An article when sold at a certain price results in 10% loss. If the selling price is increased by Rs. 100, then the profit percent is 10%. What is the costs price of the article?

    (a) Rs. 400 (b) Rs. 500 (c) Rs. 600 (d) Rs. 800

29. Two Honda bikes are sold at same price, one at a loss of 25% and other at a profit of 25%. What is the profit or loss in the whole transaction?

    (a) Loss 6.25% (b) 0% (c) Profit 6.25% (d) Loss 10%

30. A shop owner buys 30 candles and sells 20 out of it at a profit of 30% and remaining at a loss of 15%. What is his profit percentage in the complete deal?

    (a) 15% (b) 20% (c) 7.5% (d) 12.5%

31. A shop owner while selling marbles incurs a loss of 5% on one-fourth of the lot and loss of 10% on the next one-fourth of the lot too. How much higher should he sell the remaining goods to get an overall profit of 10%?

    (a) 37.5% (b) 27.5% (c) 25% (d) 15%

32. Vikas, a shrewd shop-keeper buys goods at $9/10^{th}$ of the MRP and while selling sells them at a price 8% higher than the MRP. His profit percentage is:

    (a) 18% (b) 19% (c) 20% (d) 21%

33. While selling a chair the seller keeps the marked price higher by 20%. But on haggling with the customer he lowers the marked price by 20%. What will be his profit/loss percentage?

    (a) 0% (b) –1% (c) –4% (d) –2.2%

34. A shopkeeper marks up his goods by 25%. What is the maximum discount that he can offer so that he does not make any loss?

    (a) 16.66% (b) 20% (c) 25% (d) 33.33%

35. A merchant marks his merchandise by 20% and offers a discount of 5% to gain trust of the public. If his selling price is Rs. 456, what is the marked price?

    (a) Rs. 400 (b) Rs. 420 (c) Rs. 470 (d) Rs. 480

36. C.C.D being favorite hangout place for youth offers three different successive discount schemes. You being the youth of the nation which one will you select?

    Scheme I: Discounts of 10%, 20%, 5%.

    Scheme II: Discounts of 20%,10%, 5%.

    Scheme III: Discounts of 5%, 20%,10%

    (a) I or II (b) II or III (c) I or III (d) any of I or II or III

37. After getting two successive discounts on the listed price of Rs. 150, a shirt is sold for Rs. 105. If the first discount was 12.5%, then the second discount was?

(a) 32.5% (b) 30% (c) 25% (d) 20%

38. Even after giving discount of 10% for a bulk order of paans for a shaadi, a pan-walla still makes a profit of 20%. How much higher must he have marked his paans to get this scheme working?

(a) 25% (b) 22% (c) 30% (d) 33.33%

39. The scheme "buy 4, get 1 free" is equivalent to a discount percentage of:

(a) 25% (b) 20% (c) 16.66% (d) 12.5%

40. A ration shop owner claims to be "man of the masses" and thus he sells the goods at cost price. But in reality uses 800 grams as weight instead of 1 kilogram. What will be his profit percentage on selling 2 kgs of goods?

(a) 25% (b) 50% (c) 20% (d) 40%

Averages

Whenever we speak of average, we understand that

$$\text{Average} = \frac{\text{Sum of all observations}}{\text{Total number of observations}}$$

This is more specifically called the Arithmetic Mean and would differ in interpretation and applications from Weighted Average as we will learn ahead. But for the moment it hardly matters if we use either term – Average or Arithmetic Mean.

In questions on averages, one would usually come across data of the type "the average of n persons is A...". From such a data, it should be obvious that the sum or total of all observations is $A \times n$.

Another technique worth noting before we jump headlong into examples and problems is: Consider that we are finding the average of 8 observations. If the sum (total) of the eight observations increases by 48 (because of one or more observations increasing in value), it should be obvious that the average would increase by $\frac{48}{8} = 6$. Conversely if the average changes by 2, the sum (total) of the observations should change by 2 × 8 = 16.

E.g. 1: The average weight of 10 members was 70 kgs. Find the average weight if a eleventh guy weighing 81 kgs joins the group.

Sum of weights of the 10 members = 70 × 10 = 700

Thus, sum of the weights after the eleventh guy joins the group = 700 + 81 = 781.

Thus new average = $\frac{781}{11} = 71$

While the above is straight-forward, this method may necessitate the use of pencil work if the numbers are not very manageable, as in the next example. So, we would learn a oral approach for the above and also the method is much more intellectually stimulating.

Considering that the eleventh guy also weighs same as the earlier average i.e. 70 kgs, we now have 11 kgs to be distributed equally among 11 students. Thus each will receive 1 kg and the new average is 71 kgs.

E.g. 2: The average amount that each student has is Rs. 167.75. When a student having Rs. 183.45 joins the group, the new average that each student has is now Rs. 170.89. Find the number of people originally in the group.

Using the formula, if there were n students earlier, then we would have

$$167.75 \times n + 183.45 = 170.89 \times (n+1)$$

$$\Rightarrow 3.14 \times n = 12.56$$

$$\Rightarrow n = \frac{12.56}{3.14} = 4$$

Using the "intellectually stimulating" method: Considering that the new student also has Rs. 167.75, there is now Rs. 183.45 – 167.75 = Rs. 15.70 that is distributed among all members (n + 1) and each member receives Rs. 170.89 – 167.75 = Rs. 3.14. Thus there are in all $\frac{15.70}{3.14} = 5$ members after the new student joins and there were 4 students originally in the group.

The above method is exactly same as that using equations, but it gives a better idea of how averages are formed (by distributing the entire sum equally among each member) than the equation method. Also it helps one do the calculations orally.

E.g. 3: While entering the data of marks scored by a student in six subjects, marks of one subject were wrongly entered. Instead of keying in 53 marks, 83 marks were entered. Find the increase in the average because of the error.

It should be obvious because of the earlier note that the sum (total) of marks of the six subjects would be higher by 30 marks and thus the average would be higher by $\frac{30}{6} = 5$ marks.

E.g. 4: When a member weighing 87 kgs is replaced by a member weighing 52 kgs, the average weight of the members of the group reduces by 5 kgs. Find the number of members in the group.

Because of the replacement, the sum of weights of all the members decreases by 87 – 52 = 35 gs. Because of a reduction of 35 kgs in the sum, the average reduces by 5kgs. Thus there have to be 7 members in the group.

E.g. 5: If Rahul scores 80 runs in his next innings, his average runs per innings would increase by 0.78. But if he scores only 30 runs in his next innings, his average runs per innings would decrease by 0.22. Find the number of innings played by Rahul so far.

The difference in the runs scored in the two scenarios given is 50 runs. Thus the total, after the next innings is played, would differ by 50 runs, depending on which of the two scenario happens and this difference of 50 runs would cause the average to swing by 0.78 – (–0.22) = 1 runs per innings. Thus Rahul would have played a total of 50 innings after the next innings and so he has played 49 innings so far.

Average of a A.P.

An AP is nothing but a series of numbers where the difference between consecutive numbers is the same. E.g. 17, 21, 25, 29, 34, 37, 41 is an AP where the difference between consecutive numbers is 4.

If we have to find the average of all the terms in any AP, one need not find the sum of all the terms. The average of all terms in an AP is the average of the first and last number.

Another point worth noting is that the average of an AP is the middle term. If the set has even number of natural numbers, then the average of the set is the average of the two middle numbers.

E.g. 6: Find the average of all the multiples of 3 that lie between 100 and 200.

We need to find the average of 102, 105, 108, ..., 195, 198. This is an AP and hence the average of al the terms will be $\frac{102+198}{2} = 150$

Exercise

1. The average sales for the months Jan to Mar is 40, the average sales for the months Mar to June is 50 and the average of the months Jan to June is 45. Find the sales in the month of Mar.

   a. 30 b. 40 c. 45 d. 50

2. The average of Sachin after 80 one day matches is 55. If Sachin's target is increase his average to 57 after the next one day match, how much should Sachin score in the next one day match. (Average = total run ÷ total number of innings)

   a. 213 b. 214 c. 215 d. 217

3. When a student weighing 68 kgs joins a group of 8 students, the average of the group increases by 1.5 kgs. Find the average of the original group.

   a. 53 b. 54.5 c. 56 d. 57.5

4. From a group of 12 students with average weight being 72.5 kgs, two students leave. Because the two leave, the average of the group falls to 71.25 kgs. Find the average weight of the two students who leave the group.

   a. 78.5 b. 78.75 c. 79.25 d. 157

5. If a student weighing 70 kgs joins a group of n students, the average of the group increases by 1 kgs. If the new student weighed 55 kgs, the average of the group would have declined by 2 kgs. Find n.

   a. 3 b. 4 c. 5 d. 6

6. Ten years ago, the average age of a couple was 27 years old. Today also the average age of the couple and their baby is 27 years old. Find the present age of the baby.

   a. 3 b. 4 c. 5 d. 7

7. By what does the average of all odd numbers between 50 and 130 differ from the average of all multiples of 3 between 20 and 160.

   a. 0 b. 1 c. 2 d. 3

8. The average weight of A, B and C is 45 kgs. If the average weight of A and B is 40 kgs and that of B and C is 43 kgs, find the weight of B.

   a. 0 b. 11 c. 41 d. 42

9. In the first 10 overs of a cricket match, the run rate was only 3.2 runs per over. What should be the run-rate in the remaining 40 overs to reach the target of 282 runs?

   a. 6.25 b. 6.15 c. 7.25 d. 8.50

10. The average age of a class of 30 students is 18 years. If the teacher is included, the average of the class increases to 19 years. Find the age of the teacher.

    a. 50 b. 48 c. 49 d. 47

11. The average weight of a group of four friends increases by 2.5 kgs when one of them weighing 65 kgs leaves the group. Find the average of the remaining 3 friends.

    a. 78.5 b. 75 c. 70 d. 72.5

12. The average of a group of eight friends increases by 4 years when the youngest of them is not considered and decreases by 3 years when the oldest of them is not considered. Find the difference between the ages of oldest and the youngest of them.

    a. 56 b. 49 c. 42 d. 40

Weighted Average/Alligation

Consider a class. The average age of all the boys in the class is 24 years and the average age of all the girls in the class is 20 years. What is the average age of all the students in the class?

Is it the average of 20 and 24 i.e. 22?

Not really, this is not a case of Arithmetic mean but that of Weighted Average.

If there were 20 boys and 10 girls, the sum of weights of the boys would be 24 × 20 = 480 and the sum of the ages of the girls would be 20 × 10 = 200. Thus the average weight of the entire class would be $\frac{480+200}{20+10} = \frac{680}{30} = 22.66$.

Had there been 5 boys and 15 girls, the average of the weights of all the students in the class would be $\frac{24\times 5+20\times 15}{5+15} = \frac{420}{20} = 21$.

If there were equal boys and girls in the class, say 10 each, then average would be $\frac{24\times 10+20\times 10}{10+10} = \frac{440}{20} = 22$, which was the arithmetic mean of 20 and 24.

Thus we see that the average weight of the class would depend on the number of boys and girls. If they are equal in numbers, the average is same as the arithmetic mean. If the number of boys is more than the girls, the average is more towards 24 and if the number of girls is more than the number of boys, the average gets pulled towards 20. Hence it is called Weighted Average where the two groups, boys and girls in this example, exert an influence (weight) on the average and pull the average towards their group's average.

If the average of two groups is A_1 and A_2, then the average of both the groups together will be given by the formula:

$A_{avg} = \frac{A_1\times w_1 + A_2\times w_2}{w_2 + w_2}$, where w_1 and w_2 are the weights of each group.

E.g. 1: A trader bought 8 kgs of rice costing Rs. 15 per kg and 22 kgs of rice costing Rs. 25 per kg. What is his average cost price per kg?

Using the formula, average cost price = $\frac{15\times 8+25\times 22}{8+22} = \frac{120+550}{30} = \frac{670}{30} = 22.33$ per kg.

The biggest problem that few students have in these types of problem is identifying which values to take as A_1 and A_2 and which values to take as weights. To overcome this difficulty, few pointers are:

1. Ask yourself the question: "What are we finding the average of?" Are we finding the average of the cost prices or of the kgs? Since we are finding the average of the cost prices, A_1 and A_2 refer to the cost price.

2. The numerator $A_1\times w_1 + A_2\times w_2$ always has some meaning. In this example 15 × 8 is the amount spend on first variety of rice and 25 × 22 is the amount spent on the second variety. Thus $15\times 8+25\times 22$ is the total amount spent in Rs. Now what would you divide the total amount spent with? The total kgs (8 + 22) or the sum of cost per kgs (15 + 25)? Since we need Rs/kg, we would divide it with total weight and hence the denominator $w_1 + w_2$ should refer to the kgs.

E.g. 2: 9 litres of milk and water solution having 40% milk is mixed with 3 litres of milk and water solution having 70% milk. Find the percentage of milk in the mixture of the two solutions.

Are we finding the average of the percentage of milk or are we finding the average of litres?

If the answer to the above is not very clear, try thinking what significance does 40% × 9 or 70% × 3 have. The first solution has 40% milk in it and we have 9 litres of it, thus 40% × 9 would be the amount of milk in the solution. Thus 40% × 9 + 70% × 3 refers to the total amount of milk in the mixture. What would we divide the total amount of milk in the mixture with: total volume (9 + 3) or sum of the percentage of milk (40% + 90%). To find the percentage of milk, one needs to divide the total amount of milk with the total volume.

The percentage of milk in the mixture = $\frac{40 \times 9 + 70 \times 3}{9 + 3} = \frac{360 + 210}{12} = \frac{570}{12} = 47.5\%$

E.g. 3: One-third of the goods are sold at a profit of 10%. At what profit percent should the rest of the goods be sold so as to earn an overall profit of 20%?

While this question can be solved using equations or by assuming the amount of goods (Rs. 300 is a good assumption), it can also be solved by considering it a case of weighted average. In this case we are considering the average of the profit percentages. There are two parts, one at a profit percentage of 10% and other at, say *x*% such that the average profit percentage of the two parts is 20%.

$$\frac{10 \times \frac{1}{3} + x \times \frac{2}{3}}{\frac{1}{3} + \frac{2}{3}} = 20$$

$$10 \times \frac{1}{3} + x \times \frac{2}{3} = 20 \Rightarrow 10 + 2x = 60 \Rightarrow x = 25\%$$

Exercise

1. Two types of oil having the rates of Rs. 4/kg and Rs. 5/kg are mixed in order to produce a mixture having the rate Rs. 4.6/kg. What should be the amount of the second type of oil if the amount of the first type is 40 liters?

   a. 60 lts b. 20 lts c. 50 lts d. 10 lts

2. How many kilograms of sugar worth Rs 3.6/kg should be mixed with 8 kg of sugar worth Rs. 4.20/kg such that by selling the mixture at Rs. 4.4/kg there may be a profit of 10%?

   a. 4 kgs b. 8 kgs c. 2 kgs d. 6 kgs

3. A man purchased a cow and a calf for Rs. 1300. He sold the calf at a profit of 20% and the cow at a profit of 25%. His total profit was 23%. Find the cost price of the cow.

   a. Rs. 520 b. Rs. 780 c. Rs. 800 d. Rs. 500

4. Ramesh sells 1/3 of his goods at 20% profit and the remaining at 10% loss, what would be his over all profit or loss percentage?

   a. 0% b. 10% c. 5% d. –10%

5. In what ratio should water (available free of cost) be mixed with soda costing Rs 12 per liter so as to make a profit of 25% by selling the diluted liquid at Rs. 13.75 per liter?

   a. 11 : 1 b. 1 : 11 c. 1 : 12 d. 12 : 1

Alligation:

Consider the question: In what ratio of weight should rice costing Rs. 15/kg be mixed with rice costing Rs. 25/kg, so that the mixture is worth Rs. 17.5/kg?

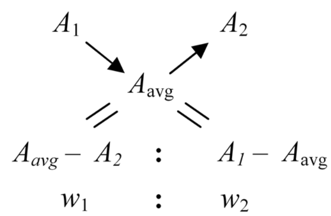
In this question, where A_1, A_2 and A_{avg} is given and the ratio of the weights $w_1 : w_2$ is asked, while we can solve the questions using the weighted average formula, a shorter technique, called Alligation exists.

Using weighted average formula, $\frac{15 \times w_1 + 25 \times w_2}{w_2 + w_2} = 17.5$.

Cross-multiplying and rearranging gives us, $7.5 \times w_2 = 2.5 \times w_1 \Rightarrow \frac{w_1}{w_2} = \frac{7.5}{2.5} = \frac{3}{1}$

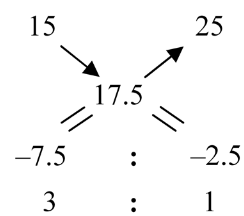
The Alligation Method:

This is a visual method where A_1 and A_2 are written in a row, slightly separated from each other and A_{avg} is written between them, slightly lower. Then the following differences $A_1 - A_{avg}$ is written diagonally across and $A_{avg} - A_2$ is written again diagonally across. The ratio of $A_{avg} - A_2 : A_1 - A_{avg}$ is the ratio of the weights, $w_1 : w_2$.



The ratio of weights is $\frac{w_1}{w_2} = \frac{A_2 - A_{avg}}{A_{avg} - A_1}$ (This formula is just a re-arrangement of the weighted average formula)

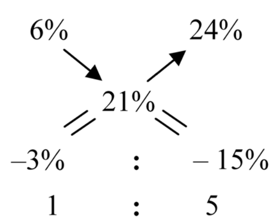
Thus in this question, the answer can be got straight from:



Please note that Alligation is easy only when A_{avg} is given. If it is not given and is required to be found, working with the weighted average formula works best.

E.g. 4: Part of goods worth Rs. 15,000 were sold at a profit of 6% and rest at a profit of 24%. If the overall profit on the entire amount is 21%, goods worth how much was sold at 6% profit?

Since the average profit percentage (on both the parts combined) is given, we will use Alligation.



Thus the ratio of goods sold at 6% profit and 24% profit is 3 : 15 i.e. 1 : 5 i.e. $\frac{1}{6}^{th}$ of the total are sold at 6% profit.

Thus required answer is $\frac{1}{6} \times 15000 = 2500$.

Using weighted average: $\frac{6\% \times x + 24\% \times (15000 - x)}{x + (15000 - x)} = 21\%$

i.e. $6x + 24 \times 15000 - 24x = 21 \times 15000$
i.e. $18x = 3 \times 15000$ i.e. $x = 2,500$

Application of Weighted Average/Alligation in Mixtures:

While we have already solved quite a few questions based on mixtures, some aspects remain to be looked at when we are dealing with solutions of milk and water.

1. Identifying A_1, A_2 and A_{avg}

When we are mixing two solutions of milk and water, A_1 and A_2 refer to the percentage of milk in each of the solution being mixed and A_{avg} refer to the percentage of milk in the mixture of the two solutions.

When the numeric value of percentage of milk is not a manageable number, as in a solution having milk and water in the ratio of 3 : 4, one should be using the proportion of milk (or water) i.e. $\frac{\text{amount of milk (or water)}}{\text{total volume}}$. Under no situation should one use the ratio $\frac{\text{milk}}{\text{water}}$.

2. Should we work with percentage of milk or water?

It does not matter what you work with – percentage or proportion of milk or water. Just remember that you should be uniform in what you use. So if you are using proportion of milk in the first solution, you have to use proportion of milk in the second solution and you will get the proportion of milk in the resultant mixture.

3. Identifying w_1, w_2

The weights in this situation refer to the volume of the solutions. If the question has a data like: "Two solutions of milk and water are mixed in the ratio 4 : 5...", w_1 could be taken as 4 and w_2 as 5.

4. When one solution is pure milk (or pure water)

In few questions, pure water or pure milk will be added to a solution of milk and water. In such cases, remember that percentage of milk in pure water is 0% (proportion of milk in pure water is 0) and percentage of milk in pure milk is 100% (proportion of milk in pure milk is 1)

E.g. 5: Two solutions having ratio of milk and water in the ratio 2 : 3 and 1 : 5 are mixed in ratio of volumes of 3 : 5. Find the ratio of milk and water in the resultant mixture.

Since A_{avg} is asked and not given, we will use the formula for weighted average. (Alligation would not work very well in this case, try it for yourself).

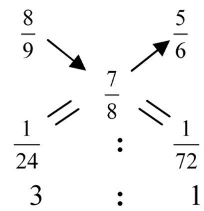
Working on proportion of milk, we have

$$A_{avg} = \frac{\frac{2}{5}\times 3 + \frac{1}{6}\times 5}{3+5} = \frac{\frac{6}{5}+\frac{5}{6}}{15} = \frac{36+25}{30\times 15} = \frac{61}{450}$$

Remember that $\frac{61}{450}$ is the proportion of milk $\left(\frac{\text{amount of milk}}{\text{total volume}}\right)$ in the resultant mixture. Thus the ratio of milk to water is 61 : (450 – 61) i.e. 61 : 389

E.g. 6: How many litres of solution having milk and water in ratio 8 : 1 should be mixed with 12 litres of solution having ratio of milk and water in the ratio 5 : 1 so that the resultant mixture has milk and water in the ratio 7 : 1

Since the proportion of milk in the resultant solution is given, we will use Alligation.



Thus the two solutions are mixed in the ratio of $\frac{1}{24} : \frac{1}{72}$ i.e. 3 : 1.

Since there are 12 litres of the second solution, we would need to add 36 litres of the first solution.

Using Weighted Averages Formula:

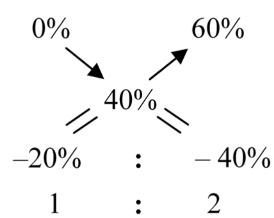
$$\frac{\frac{8}{9} \times x + \frac{5}{6} \times 12}{x + 12} = \frac{7}{8}$$

i.e. $\frac{8}{9} \times x + 10 = \frac{7}{8} \times x + \frac{7}{8} \times 12 \Rightarrow \frac{1}{72} x = \frac{1}{2}$ i.e. $x = 36$.

E.g. 7: How many litres of water needs to be added to 10 litres of a solution having milk and water in the ratio 3 : 2 to result in a solution having milk and water in the ratio 2 : 3?

Since the proportion of milk and water in the resultant solution after mixing, A_{avg} is given to be $\frac{2}{5}$, we will use Alligation.

Also in this case rather working on proportion of milk, its smarter to work with percentage of milk as the numbers are very manageable $\left(\frac{3}{5} = 60\%, \frac{2}{5} = 40\%\right)$.



Thus the two solutions are mixed in the ratio of 20 : 40 i.e. 1 : 2.

Since we have 10 litres of 60% milk solution, we would need to add 5 litres of water.

Using Weighted Average Formula:

Working with milk proportion, $\frac{0 \times x + \frac{3}{5} \times 10}{x + 10} = \frac{2}{5}$ i.e. $6 = 0.4x + 4$ i.e. $x = 5$

Working with water proportion, $\frac{1 \times x + \frac{2}{5} \times 10}{x + 10} = \frac{3}{5}$ i.e. $x + 4 = 0.6x + 6$ i.e. $0.4x = 2$ i.e. $x = 5$

Exercise

6. A mixture of 125 gallons of wine and water containing 20% water. How much water should be added to the mixture in order to increase the percentage of water to 25% of the new mixture?

   a. 25 gallons b. 12.5 gallons c. 8.33 gallons d. 6.25 gallons

7. A solution having milk and water in the ration 2 : 3 is mixed with another solution having milk and water ratio 3 : 2 , the resultant solution has milk and water in the ratio 1 : 1, in what ratio were the two solutions mixed?

   a. 1 : 1 b. 1 : 2 c. 1 : 3 d. 2 : 1

8. A solution having milk and water in the ration 2 : 5 is mixed with another solution having the milk and water ration 5 : 2 the resultant solution has milk and water in equal quantities. In what ratio were the two solutions mixed?

   a. 2 : 1 b. 1 : 2 c. 1 : 1 d. 3 : 1

9. A solution having milk and water in the ratio 2 : 3 is mixed with another milk and water solution. The resultant solution has milk and water in the ratio 5 : 4. Find the milk and water ratio in the second solution if the two solutions were mixed in equal quantities.

   a. 1: 1 b. 32 : 13 c. 47 : 43 d. 3 : 7

10. A solution having milk and water in the ratio 2 : 3 is mixed with another milk and water solution. The resultant solution has milk and water in the ratio 4 : 5. Find the milk and water ratio in the second solution if the two solutions were mixed in the ratio 3 : 2

    a. 21 : 23 b. 22 : 23 c. 23 : 22 d. 23 : 21

Removal and replacing part of a solution with pure solution

A standard question in many entrance exams goes as follows:

8 litres are drawn out from 80 litres of a solution having milk and water in the ratio 7 : 2 and is replenished with water. Again litres of solution is drawn out and replenished with water. Find the ratio of milk and water in the solution now.

In these types of questions when a part of a solution is drawn out and replaced with a pure part (water in this example), follow the 2 steps given below:

1. Work on that part (milk or water) that is NOT replaced after removal. Since we are replacing the drawn out solution by water, we have to work on milk in this example

2. There are three quantities that are related, viz.

a. Initial proportion (or percentage) of part not replaced

b. Final proportion (or percentage) of part not replaced

c. Fraction of the solution being removed and replaced. In this case it is 8 litres out of 80 litres i.e. $\frac{8}{80}$ *i.e.* $\frac{1}{10}^{th}$. In the formula which is going to be given in a moment, we will have to work on the fraction that is left behind after removal i.e. $\frac{9}{10}^{th}$ in this example.

The three quantities are related as follows:

Final proportion = Initial proportion × (fraction left after removal)^n

Or

Final proportion = Initial proportion × (1 – fraction removed)^n

n is the number of times the removal and replacement is carried out.

Two out of the three quantities will be given and third will be asked.

In this example, final proportion of milk = $\frac{7}{9}\times\frac{9}{10}\times\frac{9}{10}=\frac{63}{100}$.

Thus ratio of milk to water is 63 : 37

E.g. 8: 16 litres are removed from 80 litres of a solution of milk and water and replaced with milk. The ratio of milk and water now is 5 : 1. Find the ratio of milk and water in the initial solution.

Do not forgot to identify what part are you going to work with. The formula given does not work with both milk or water. It only works with that part that is NOT replaced, water in this example.

Fraction removed = $\frac{16}{80}$ i.e. $\frac{1}{5}$. Thus, fraction left after removal = $\frac{4}{5}$

In this case the final proportion of water is given as $\frac{1}{6}$ and thus we have to find the initial proportion.

$\frac{1}{6}$ = initial proportion $\times\frac{4}{5}$

Thus initial proportion of water = $\frac{5}{24}$ and the required ratio of milk and water is 19 : 5

E.g. 9: 12 litres of a solution having milk and water in the ratio 4 : 5 are drawn out and replaced with water. This operation is done once more. If the ratio of milk and water now is 5 : 4, find the volume of the original solution.

In this case we have to work on proportion of milk,

We know 12 litres are drawn out. But we do not know the original volume and hence the fraction removed is unknown. Both the initial and final proportion of milk is given. Thus,

$$\frac{5}{9} = \frac{4}{5} \times (\text{fraction left after removal})^2$$

$$\Rightarrow (\text{fraction left after removal})^2 = \frac{25}{36}$$

fraction left after removal $= \frac{5}{6}$, i.e. fraction removed is $\frac{1}{6}$.

But we know that 12 litres is drawn out. So 12 litres must be $\frac{1}{6}^{th}$ of the total volume.

Thus, total volume = 6 × 12 = 72 litres.

Exercise

11. From a solution which has milk and water in the ration 3 : 2, 20% of the solution is removed and replaced with water. What will be the ratio of milk and water in the resultant solution?

    a. 13 : 12   b. 12 : 13   c. 1 : 1   d. 1 : 2

12. From a solution which has milk and water in the ration 3 : 1, 10% of the solution is removed and replaced with milk. What will be the ratio of milk and water in the resultant solution?

    a. 31 : 9   b. 9 : 31   c. 27 : 13   d. 13 : 27

13. How many litres should be removed out of 48 litres of solution of milk and water and be replaced with equal amount of water so that the proportion of milk drops from 80% to 75%.

    a. 2 lts   b. 3 lts   c. 10 lts   d. 15 lts

14. When 20% of a solution of milk and water is removed and replaced with water, the ratio of milk and water becomes 2 : 3. Find the ratio of milk and water in the original solution.

    a. 1 : 2   b. 2 : 1   c. 1 : 1   d. 1 : 3

15. From a solution having milk and water in the ratio 2 : 3, 15 lts of solution is drawn out and replaced with water. This operation is done a total of thrice. If the ratio of milk and water now is 25 : 231, find the volume of original solution.

    a. 20 lts   b. 40 lts   c. 30 lts   d. 50 lts

Assignment - Averages & Weighted Averages

1. The average sales for the months January to March is 40, the average sales for the months March to June is 50 and the average of the months January to June is 45. Find the sales in the month of March.

   (a) 30 (b) 40 (c) 45 (d) 50

2. A class teacher buys caps for boys and girls in his class. There are 20 boys and 10 girls in the class. A cap for a boy costs Rs. 15 and for a girl costs Rs. 30. But the teacher wants everyone to pay equally. How much does every student pay?

   (a) 30 (b) 25 (c) 20 (d) 18

3. The average age of women at a tea party is 58. When three women leave early and two join after those three have left, the average age of the women becomes 60. What is the difference between the sum of the ages of the group of three who left and that of the group of two who joined?

   (a) 30 (b) 28 (c) 14 (d) 15

4. Ramprasad wanted to find average of all the prime numbers between 10 and 20. But erroneously he used a number N instead of 19, because of which he got the average as 18. What is the value of N – 19?

   (a) 31 (b) 12 (c) –12 (d) 3

5. Nawab Pataudi had an average of 63 runs in first 15 matches and after having a good batting streak in next two matches, he increased the average to 65. What is his average in the last two matches?

   (a) 75 (b) 80 (c) 34 (d) 40

6. What is the difference between average of first 50 odd numbers and average of first 51 even numbers?

   (a) 2 (b) 0 (c) 1 (d) 3

7. Among three numbers, the first is twice the second and second is twice the third. If the average of the reciprocals of the three numbers is 7/12, the numbers are:

   (a) 16, 8, 4 (b) 20, 10, 5 (c) 24, 12, 6 (d) 4, 2, 1

8. When a student weighing 68 kgs joins a group of 8 students, the average of the group increases by 1.5 kgs. Find the average of the original group.

   (a) 53 (b) 54.5 (c) 56 (d) 57.5

9. If a student weighing 70 kgs joins a group of n students, the average of the group increases by 1 kgs. If the new student weighed 55 kgs, the average of the group would have declined by 2 kgs. Find n.

   (a) 3 (b) 4 (c) 5 (d) 6

10. Ten years ago, the average age of a couple was 27 years old. Today also the average age of the couple and their baby is 27 years old. Find the present age of the baby.

    (a) 4 (b) 5 (c) 6 (d) 7

11. In what ratio should be rice worth Rs. 20/kg be mixed with rice worth Rs. 15/kg to get rice worth Rs. 16.50/kg

    (a) 3 : 7 (b) 7 : 3 (c) 5 : 1 (d) 2 : 5

12. How many litres of solutions having 30% milk and 70% milk should be mixed so that we get 40 litres of 45% milk solution?

    (a) 10 , 30 (b) 12 , 28 (c) 15 , 25 (d) 25 , 15

13. A solution contains A and B in ratio 3 : 4. A second solution contains A and B in ratio 2 : 3. If 14 ltrs of first solution are mixed with 15 ltrs of second solution, find the ratio of A and B in the final mixture.

(a) 5/7 (b) 17/29 (c) 12/17 (d) 8/5

14. An oil merchant bought some 1200 litres of oil. Part of the stock was sold at 5% loss and the remaining part at 7% profit. If he made an overall profit of 2%, how many litres of oil was sold at a loss?

(a) 700 (b) 500 (c) 600 (d) 800

15. Contents of three vessels having volumes in the ratio of 2 : 3 : 4 and having solutions with milk concentrations of 40%, 30% and 20% are mixed together in a fourth large vessel. Find the concentration of milk after the mixing.

(a) 30 % (b) 27.7 % (c) 27.2 % (d) 26.1 %

16. How many litres of water needs to be added to 25 litres of a solution having milk and water in the ratio 8 : 5, such that the resultant has milk and water in the ratio 5 : 8.

(a) 15 (b) 20 (c) 25 (d) 30

17. 12 litres of milk is added to 30 litres of solution having milk and water in the ratio 3 : 5. Find the ratio of milk and water after the addition.

(a) 22 : 25 (b) 34 : 25 (c) 9 : 25 (d) 31 : 25

18. A 50 kg alloy of tin and lead has 60% lead in it. Some extra tin is added to make the percentage of tin in the alloy equal to 60%. What is the total weight of the alloy now?

(a) 80 kg (b) 100 kg (c) 75 kg (d) 60 kg

19. A bartender replaces 10 ltrs of wine with water in a cask full of wine. He again takes out 10 ltrs of mixture and replaces it with water. Now the ratio of wine to water in the solution is 9 : 16. What is the volume of the cask?

(a) 40 ltrs (b) 25 ltrs (c) 15 ltrs (d) 30 ltrs

20. A jar containing a mixture of milk and water has 40% milk. A part of this solution is replaced by another mixture of milk and water having 19% milk. If now the mixture has 26% milk, what fraction of the jar was replaced?

(a) 1/3 (b) 2/3 (c) 2/5 (d) 3/5

Time, Speed, Distance

Relationship between Time, Speed and Distance

Speed is the rate at which distance is covered i.e. how much distance is covered in a unit of time. So we get the formula:

$$\text{Speed} = \frac{\text{Distance}}{\text{Time}}$$

Since Distance is measured in km or meters, and Time is measured in hours, minutes or seconds, so Speed is measured in kmph or m/s.

Conversion from kmph to m/s and vice versa

$$1 \text{ kmph} = \frac{1 \text{ km}}{1 \text{ hr}} = \frac{1000 \text{ m}}{3600 \text{ s}} = \frac{5 \text{ m}}{18 \text{ s}} = \frac{5}{18} \text{ m/s}$$

Hence to convert a speed, which is in kmph to m/s, just multiply it by $\frac{5}{18}$. On the other hand, to convert a speed in m/s to kmph, divide it by $\frac{5}{18}$ or in other words, multiply it by $\frac{18}{5}$.

Using the above, we see 18 kmph = 5 m/s

36 kmph = 10 m/s

54 kmph = 15 m/s

72 kmph = 20 m/s and so on …

E.g. 1: A cyclist covers a distance of 750 m in 2 min 30 sec. What is the speed in kmph of the cyclist?

2 min 30 sec is equal to 150 sec.

Using the formula given above, we get, Speed = 750/150= 5 m/s

To convert to kmph, $5 \times \frac{18}{5} = 18$ kmph

E.g. 2: Which of the following trains is the fastest?

(a) 25 m/s (b) 1500 m/min (c) 90 kmph

We need to convert each to the same unit so converting them to m/s

$1500 \text{ m/min} = \frac{1500 \text{ m}}{1 \text{ min}} = \frac{1500 \text{ m}}{60 \text{ sec}} = 25 \text{ m/s}$ 90 kmph = $90 \times \frac{5}{18} = 25$ m/s

So all are the same speed.

E.g. 3: A man in a train notices that he can count 21 telephone posts in one minute. If they are known to be 50 m apart, then at what speed is the train traveling?

Between 21 telephone posts, there will be 20 stretches of 50 m each. Hence the total distance will be 20 × 50 = 1000 m

Total time taken to cover this stretch = 1 min = 60 s.

Hence Speed of the train = 1000/60 m/s = 16.66 m/s

E.g. 4: If a man walks at the rate of 5 kmph, he misses a train by 7 minutes. However, if he walks at the rate of 6 kmph, he reaches the station 5 minutes before the arrival of the train. Find the distance covered by him to reach the station.

Difference in the two times = 12 min

Say, at a speed of 5 kmph, he takes t mins, so distance = $5 \times \frac{t}{60}$

Then at a speed of 6 kmph he will take (t – 12) min, so distance = $6 \times \frac{t-12}{60}$

The distance covered in the two cases is same so $5 \times \frac{t}{60} = 6 \times \frac{t-12}{60}$

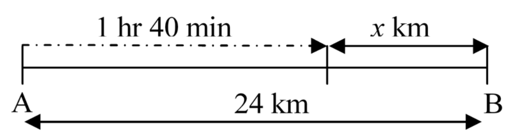
So we get distance covered by him to reach the station = $6 \times \frac{72-12}{60} = 6$ km

E.g. 5: How many minutes does Suman take to cover a distance of 400 m if she runs at a speed of 20 kmph?

Suman's Speed = $20 \times \frac{5}{18} = \frac{50}{9}$ m/s

Time taken to cover 400 m = $400 \times \frac{9}{50}$ = 72 s = 1.2 min

E.g. 6: While covering a distance of 24 km, a man noticed that after walking for 1 hour and 40 minutes, the distance covered by him was $\frac{5}{7}^{th}$ of the remaining distance. What was his speed in m/s?



It is given that the covered distance is $\frac{5}{7}^{th}$ of the remaining distance x

So $24 - x = \frac{5}{7}x$

We get x = 14 km. So 10 km has been covered in 1 hr 40 min.

Hence Speed = $\frac{10}{1\frac{2}{3}}$ = 6 kmph

Converting to m/s, we get Speed = $6 \times \frac{5}{18}$ = 1.66 m/s

Exercise

1. A person goes from A to B travelling a distance of 30 km in 3 hrs. What is his speed in m/s?

   a. 2.78 b. 3 c. 10 d. 36

2. If a person moving at the speed of 1.8 kmph travels from point A to B in 1 day, 4 hrs, 20 minutes. Find the distance between A and B.

   a. 66 km b. 22 km c. 33 km d. 51 km

3. A man runs for 100 m at a speed of 36 kmph. How much time does he take to do it?

   a. 10 secs b. 10 mins c. 3 mins d. 60 secs

4. A is walking towards B. At one point he sees that there is a distance of 700m between B and him. He increases his speed by 75% and takes 16 secs to reach her. Had he travelled at his original speed, how much time would he have taken?

   a. 20 secs b. 24 secs c. 28 secs d. 30 secs

5. In the above question, if at the point when the distance between A and B was 700m, B had instantly moved back further 300 m, how long would it take A to reach B, if he travels at his original speed?

   a. 50 secs b. 60 secs c. 30 secs d. 40 secs

6. If instead of going at 36 kmph, Amit goes to school at 42 kmph, he reaches 10 minutes early. What is the usual time he takes to reach school and what is the distance he travels?

   a. 90 mins, 42 km b. 60 mins, 36 km c. 70 mins, 42 km d. 90 mins, 36 km

7. An express train covers the distance between stations X and Y, 1 hr faster than a normal train. Find this distance if the average speed of the express train is 80 kmph and that of the normal train is 70 kmph.

   a. 640 km b. 520 km c. 560 km d. 490 km

Proportionality between Time, Speed and Distance

This is probably the most important topic in this chapter. And also the most thought-intensive. It provides the foundation to solve many tough problems orally!

$$\text{Time} \propto \frac{1}{\text{Speed}}$$

Time is inversely proportional to speed, when distance is constant.

This is to say that, over a same distance, if the ratio of speeds is $a : b$, the ratio of the time taken will be $b : a$.

And this should be obvious, because over a same distance,

if I double my speed (ratio of speeds 1 : 2), the time taken will be half (ratio of time 2 : 1).

if I travel at $\frac{1}{3}^{rd}$ the usual speed (ratio 3 : 1), I would take thrice the time taken earlier (ratio 1 : 3)

if my speed is $\frac{3}{5}^{th}$ of usual speed (ratio 5 : 3), the time taken will be $\frac{5}{3}$ time the usual time (ratio 3 : 5).

E.g. 7: From home to office, if I travel at $\frac{3}{5}^{th}$ of my usual speed, I am late by 12 minutes. Find the time that I take usually and the time taken at the reduced speed.

In such problems, 'late by 12 minutes' implies that 12 more minutes will be taken to travel the same distance, or in other words, the difference in the time taken at the usual speed and the reduced speed will be 12 minutes.

Had my usual speed been s, the reduced speed would be $\frac{3}{5}s$. Thus the ratio of usual speed to reduced speed would be $s : \frac{3}{5}s$ i.e. $5 : 3$. (From next problem onwards, this step will be done directly).

Since time is inversely proportional to speed, the ratio of the time is 3 : 5. We also know that the difference in the time taken will be 12 minutes.

Thus we are looking for two numbers that are in the ratio 3 : 5 and the difference between them is 8 minutes. In the chapter on ratios, we learnt to assume the numbers as $3k$ and $5k$ and then we have the equation $5k - 3k = 12$ i.e. $2k = 12$ i.e. $k = 6$.

Thus the usual time taken is 3 × 12 = 18 minutes and the time taken at reduced speed is 5 × 6 = 30 minutes.

E.g. 8: E.g. If a man walks at the rate of 5 kmph, he misses a train by 7 minutes. However, if he walks at the rate of 6 kmph, he reaches the station 5 minutes before the departure of the train. Find the distance to the station.

Missing the train by 7 minutes and reaching early by 5 minutes implies that the time taken at speed of 6 kmph is 12 minutes less than the time taken at speed of 5 kmph.

The ratio of speeds is 5 : 6 and since distance is constant the ratio of the time taken will be in the ratio 6 : 5. Also as explained the difference in the time taken in the two cases differ by 12 minutes. Thus, $6k - 5k = 12$ i.e. $k = 12$.

Thus time taken at 6 kmph is 5 × 12 = 60 minutes and at 5 kmph is 6 × 12 = 72 minutes. Either of these speed and time combination can be used to find the distance. Since 60 minutes is 1 hour, its easier to find the distance as 6 kmph × 1 hr = 6 km. Check that the same distance is found using the other combination of 5 kmph and 72 minutes. Thus the distance is 6 km.

Distance α Speed

Distance is directly proportional to speed, when the time is same

This is to say, if time is constant and the ratio of speeds is $a : b$, the ratio of the distance covered will also be $a : b$.

The most common case of time remaining same would be when two persons, trains or objects start from two points simultaneously and meet each other. In this case the time that the two objects are travelling from starting simultaneously to the time they meet each other is the same for both the objects. So they will cover distances in proportion to their speeds.

E. g. 9: Two trains start simultaneously, one from Bombay to Kolkata and other from Kolkata to Bombay. They meet each other at Nagpur which is at a distance of 700 kms from Bombay. If the distance between Bombay and Kolkata is 1600 km, find the ratio of their speeds.

Since the trains started simultaneously, the time they have been travelling till the time they meet is equal. And hence the distance they cover will be in ratio of their speed. Since the train from Bombay has covered 700 km and the train from Kolkata has covered 1600 – 700 = 900 kms, the ratio of their speeds will be 700 : 900 i.e. 7 : 9.

E.g. 10: Two friends start walking towards each other with speeds in the ratio 3 : 4. When they meet it is found that the faster of them has covered 25 meters more than the slower. Find the distance that separated them initially if they are walking in opposite directions.

Again in this case since they are walking for the same amount of time the ratio of the distance covered will also be 3 : 4. The actual distance given in the question, 25 meters, refers to the difference in the distances covered by them. Thus, using the funda of ratios, $4k - 3k = 25$ i.e. $k = 25$. Thus, individually they have travelled 3 × 25 = 75 meters and 4 × 25 = 100 meters

Since the two friends are walking in opposite directions towards each other, the distance that separated them initially would have been the sum of the distances the two walked. i.e. 175 meters.

E.g. 11: A police-man starts chasing a thief. The ratio of the speeds of the thief and the policeman is in the ratio 9 : 11 and when the policeman catches the thief it is found that the policeman has covered 60 meters more than the thief. How much distance did the police have to run to nab the thief?

Since the chase starts with both of them running simultaneously, from this point onwards to the time the police has caught the thief, they are running for same duration. Thus the distance covered will be proportional to their speeds. So we are searching for two distances in the ratio 9 : 11 and the difference being 60 meters.

Assuming the distances as $9k$ and $11k$, we have $11k - 9k = 60$ i.e. $2k = 60$ i.e. $k = 30$. And the distances run by the police = 11 × 30 = 330 meters.

E.g. 12: In the movie Ghulam, Aamir is able to spot the approaching train when it is 2 km away. He has to run towards the train and reach the red kerchief hung on a pole 400 meters away from him before the train reaches the pole. How fast must Aamir run if the speed of the train is 36 kmph so that he just manages to reach the kerchief at the same time as the train reaches it.

In the same time in which Aamir runs 400 meters, the train runs 1600 meters. Thus the ratio of the speed of Aamir and the train has to be 1 : 4. Since the speed of the train is 36 kmph, Aamir should run at a speed of 9 kmph.

E.g. 13: In a race of 100 meters, A beats B by 10 meters and C by 20 meters. By how many meters does B beat C in the same race?

Most of the problem on race are based on the proportionality of speed and distance because in the case of race, the runners start running simultaneously and thus at any point of time (when the runners are yet running), they have been running for the same time.

When A has run 100 meters, B would have run 90 meters and C would have run 80 meters. Since this has happened in the same time, the ratio of the speeds of B and C is 9 : 8. Now we have to find the difference between B and C when B has run 100 meters. $9k = 100$ implies that $8k$ will be $8 \times \frac{100}{9}$ = 88.88. Hence the distance by which B wins over C is 100 – 88.88 = 11.11

Exercise

8. Two trains start simultaneously from A and B heading towards B and A respectively. If the distances they cover when they meet is 300 m and 700 m respectively, what is the ratio of their speeds?

   a. 3 : 7 b. 3 : 10 c. 7 : 10 d. 4 : 10

9. Three friends jog on a straight path everyday. To cover 500 m, first one takes 50 s, second takes 25 s and third takes 20 s. What is the ratio of their speeds?

   a. 10 : 4 : 5 b. 4 : 10 : 5 c. 4 : 5 : 10 d. 2 : 4 : 5

10. By walking at $3/4^{th}$ of his usual speed, a man reaches office 20 minutes later than usual. What is his usual time?

    a. 80 mins b. 60 mins c. 40 mins d. 30 mins

11. When the speed at which a certain distance is travelled reduces by 2 m/s, time taken increases by 10 s and when the speed reduces by 3 m/s, the time increases by 18 s. Find the distance.

    a. 120 m b. 180 m c. 240 m d. 300 m

12. Arnav arrives at his office 20 min late everyday. One day he injures his leg so his speed reduces by 25% because of which he reaches 45 min late. How much time would he take to reach his office if he decides to be on time?

    a. 75 mins b. 100 mins c. 55 mins d. 50 mins

Average Speed

During a journey, different parts of the journey could be covered at different speeds. Average speed is that uniform speed at which the total distance could be covered in the same time as it took in the actual case.

Thus, $\text{Average Speed} = \dfrac{\text{Total Distance}}{\text{Total Time}}$

It is not necessarily the average of different speeds, but can also possibly be, as seen in the following examples.

E.g. 14: Atul covered 20 kms at a speed of 30 kmph and next 30 kms at a speed of 90 kmph. Find his average speed.

The total distance that Atul covered was 20 + 30 = 50 kms. To find the average speed, we would also need to find the total time take. This can be found by adding the individual time taken over the two stretches.

Total time taken $= \dfrac{20}{30} + \dfrac{30}{90} = \dfrac{2}{3} + \dfrac{1}{3}$ = 1 hr. Thus average speed = $\dfrac{50 \text{ km}}{1 \text{ hr}}$ = 50 kmph

Please note that 50 kmph is not the arithmetic mean of 30 kmph and 90 kmph. Whereas in the following example the average speed turns out to be the arithmetic mean...

E.g. 15: Atul covered one fourth of a distance at a speed of 30 kmph and rest of the distance at a speed of 90 kmph. Find his average speed.

Since the distance is broken into one-fourth and three-fourth, let's assume the total distance to be $4d$

To find the average speed, we would also need to find the total time take. This can be found by adding the individual time taken over the two stretches.

Total time taken = $\dfrac{d}{30} + \dfrac{3d}{90} = \dfrac{6d}{90} = \dfrac{d}{15}$ hrs. Thus average speed = $\dfrac{4d \text{ km}}{d/15 \text{ hr}}$ = 60 kmph

In this case the average speed 60 kmph is the arithmetic mean of 30 kmph and 90 kmph.

Thus the average speed need not necessarily be equal to the arithmetic mean of the different speeds. But it can also be equal to the arithmetic mean. Later we will decipher when is it going to be the arithmetic mean.

E.g. 16: Atul travelled for one fourth of the total time of a journey at a speed of 30 kmph and rest of the time at a speed of 90 kmph. Find his average speed.

In this case the journey is broken into stretches of time, one-fourth of time and three-fourth of time. So let us assume the total time as $4t$.

To find the average speed, we would also need to find the total distance covered. This can be found by adding the individual distances covered over the two stretches.

Total distance covered = $30 \times t + 90 \times 3t = 300t$

Thus, average speed $\frac{300t}{4t} = 75$

Please compare e.g. 15 and 16, while the numbers used were exactly similar, the average speed are different. Also in the earlier example distance was assumed as $4d$ and in the latter one time was assumed as $4t$. These two examples basically cover most of the types of the questions in this topic.

Stretches in terms of distance or time

As seen in the above solved examples, when "stretches" of the journey are traveled at different speeds, the "stretches" could be expressed in terms of distance or in terms of time:

"Atul traveled $\frac{1}{3}^{rd}$ of a distance at a speed of 30 kmph and the rest of the distance at a speed of 60 kmph"

is different from the data "Atul traveled for $\frac{1}{3}^{rd}$ of the time at a speed of 30 kmph and the rest of the time at a speed of 60 kmph"

And the expression to find the average speed visually appears different but is essentially $\frac{\text{Total Distance}}{\text{Total Time}}$

Stretches in terms of distance:

If $d_1, d_2, d_3, \ldots$ are the distances run at speeds $s_1, s_2, s_3, \ldots.$ respectively,

Average speed = $\frac{d_1 + d_2 + d_3 + \ldots\ldots}{\frac{d_1}{s_1} + \frac{d_2}{s_2} + \frac{d_3}{s_3} + \ldots\ldots}$

The numerator is the total distance covered and denominator is the total time taken and is found by adding the time taken in each stretch.

Stretches in terms of time:

If one travels for $t_1, t_2, t_3, \ldots$ hours at speeds $s_1, s_2, s_3, \ldots.$ respectively,

Average speed = $\frac{(s_1 \times t_1) + (s_2 \times t_2) + (s_3 \times t_3) + \ldots\ldots}{t_1 + t_2 + t_3 + \ldots\ldots}$

The numerator is the total distance traveled and is found by adding the distances traveled in each time interval. The denominator is the total time taken.

E.g. 17: A man covers equal distances at a speed of 30 kmph and 60 kmph. Find his average speed.

Since the distances are equal, assuming them to be d, the average speed = $\frac{2d}{\frac{d}{30}+\frac{d}{60}} = \frac{2d \times 60}{3d} =$

40.

Special Case: Two stretches of equal distances, Harmonic Mean

A special case is when two equal distances are traveled at two different speeds, say u and v. In this case, the average speed is $\frac{d+d}{\frac{d}{u}+\frac{d}{v}} = \frac{2}{\frac{1}{u}+\frac{1}{v}} = \frac{2uv}{u+v}$

Remember to use the formula only when two equal distances are run at speeds of u and v.

E. g. 18: A man travels from a village to a post office at the rate of 25 kmph and walked back at the rate of 4 kmph. If the whole journey took 5 hours 48 minutes, find the distance of the post office from the village.

Average Speed if distance covered is same = $\frac{2ab}{a+b} = \frac{2 \times 25 \times 4}{25+4} = \frac{200}{29}$ kmph

Total distance covered in the whole journey = Average Speed × Total time taken

So total distance covered = $\frac{200}{29} \times \frac{29}{5}$ km

Hence distance from post office to village is half of this, which is 20 km.

Exercise

13. Out of a total of 100 km, A travels 30 km at 15 kmph, 28 km at 54 kmph and rest of the distance at 7 kmph. What is his average speed for the entire journey?

    a. 100/17 kmph b. 200/17 kmph c. 150/17 kmph d. 270/23 kmph

14. A child goes from his home to playground at a speed of 4 m/s. He returns at a speed of 8 m/s. What is the total time he takes for going and coming if his playground is 100 m away from home?

    a. 50 sec b. 37.5 sec c. 30 sec d. 40 sec

15. Akash travels to office at a speed of 72 kmph. He comes back at 3/4 th his original speed. If in all he takes 210 mins to cover the two distances, how far is his office?

    a. 70 km b. 84 km c. 96 km d. 108 km

16. What is the average speed of a man who travels half the distance at a speed of 25 kmph, one-third the distance at 50 kmph and the rest of the distance at 75 kmph?

    a. 450/13 kmph b. 350/13 kmph c. 450/11 kmph d. 350/11 kmph

17. What is the average speed of a man who travels half the time at a speed of 25 kmph, one-third the time at 50 kmph and the rest of the time at 75 kmph?

    a. 62.5 kmph b. 125 kmh c. 125/3 kmph d. 31.25 kmph

18. Aradhna covers a part of the journey at 20 kmph and the balance at 70 kmph taking total of 8 hours to cover the distance of 400 km. How many hours has she been driving at 20 kmph?

    a. 3.2 hours b. 5.2 hours c. 4.8 hours d. 2.8 hours

Relative Speed

The relative speed of an object is the speed of the object relative to another moving object. Suppose a bus is traveling from Mumbai to Pune. Another bus is traveling from Pune to Mumbai. When the two buses approach each other, the people sitting in them feel that the other bus is coming at a very high speed. That is because their own bus is also moving towards the other bus at a significant speed, which they are not able to access because they are themselves moving with that speed.

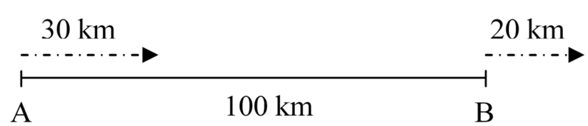
Similarly, if two cars are moving in the same direction at the same speed, the car when seen from the other car seems stationary.

Seeing from inside a moving car, the trees seem to be moving in the opposite direction.

These are all examples of relative speed.

If two objects are moving in opposite directions, either towards each other or away from each other, their relative speed will be the sum of their speeds. Say a car and a truck are traveling towards each other. Initially the distance between the two is 100 km. The car travels at a speed of 30 kmph and the truck at 20 kmph. This would imply that in 1 hr, the car would have covered 30 km and the truck would have covered 20 km thus reducing the distance between the two by 50 km. That means that effective speed of both combined would be 50 kmph. This is the relative speed of the car with respect to the truck and the relative speed of the truck with respect to the car.

On the other hand, say the car and the truck were moving in the same direction. The distance between the two was 100 km. In one hour, the car would cover 30 km and in the same time the truck would cover 20 km both in the same direction



So the distance between the two reduced by only 10 km in one hour. Their relative speed hence is 30 – 20 = 10 kmph with respect to each other.

It means that they will need 10 hrs to cover the 100 km between them.

If two objects are moving in the same direction, their relative speed will be the difference of their speeds.

If two objects are moving with the speeds of a and b respectively, then

Case (i) Moving in opposite direction, towards each other or away from each other

Relative Speed = $a + b$

Case (ii) Moving in the same direction

Relative Speed = $(a - b)$

E.g. 19: A thief is spotted by a policeman from a distance of 100 m. When the policeman starts the chase, the thief also starts running. If the speed of the thief is 8 kmph and that of the policeman 10 kmph, what is the distance that the thief would have covered before he is caught?

Since both thief and policeman are running in the same direction, their relative speed will be the difference in their speeds i.e. 10 – 8 = 2 kmph

The distance between the two was 0.1 km so this will be covered in time $\frac{0.1}{2}$ hrs

In this time, the thief would have covered a distance = $8 \times \frac{0.1}{2} = 0.4$ km = 400 m

In the question above, if the speed of the policeman is 8 kmph and the speed of the thief is 10 kmph, after how much time will the policeman catch the thief?

Here the relative speed of the policeman with respect to the thief = 8 – 10 = –2 kmph. Since the speed of the policeman is less than that of the thief, he will never be able to catch up with the thief.

E.g. 20: A train is travelling at a steady speed of 75 kmph. Currently the distance between the train and a car coming from the opposite direction towards the train is 250 km. What should be the speed of the car so that the two meet in 1 hr 15 mins?

Here the two are travelling in the opposite direction, towards each other. Their relative speed will be the sum of their individual speeds which is equal to 75 + x.

Total Distance = 250 km which they have to cover in 1 hr 15 min = 5/4 hr

Hence the relative speed required = $\frac{\text{total distance}}{\text{total time}} = \frac{250}{5/4} = 200$ kmph

Since relative speed = 200 kmph = 75 + x

Hence required speed of the car 'x' = 125 kmph

Exercise

19. A man and a woman 81 miles apart from each other, start travelling towards each other at the same time. If the man covers 5 miles per hour to the women's 4 miles per hour, how far will the woman have travelled when they meet?

    a. 50 miles b. 40 miles c. 45 miles d. 36 miles

20. Two boys begin together to build a railway track containing 352 track parts. The first boy starts laying the first part, at the rate of 10 parts an hour; and the second starts with the last part and then the second last part and so on proceeding backwards at the rate of 12 parts an hour. At which part will they meet?

    a. Between 159 & 160th b. Between 160 & 161st c. Between 190 & 191st d. Between 191 & 192nd

21. Train A travelling at 60 kmph leaves Mumbai for Delhi at 6 P.M. Train B travelling at 90 kmph also leaves Mumbai for Delhi at 9 P.M. Train C leaves Delhi for Mumbai at 9 P.M. If all three trains meet at the same time between Mumbai and Delhi, what is the speed of Train C if the distance between Delhi and Mumbai is 1260 kms?

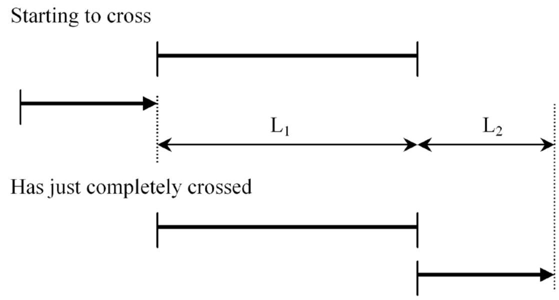
    a. 100 kmph b. 90 kmph c. 120 kmph d. 130 kmph

22. A and B leave points X and Y towards Y and X respectively simultaneously and travel on the same route. After meeting each other on the way, A takes 4 hours to reach her destination, while B takes 9 hours to reach his destination. If the speed of A is 48 kmph, what is the speed of B?

    a. 32 kmph b. 16 kmph c. 18 kmph d. 24 kmph

Trains Crossing Pole/Man/Platform/Train

When ever one object of finite length crosses another object of finite length, in crossing the object completely it covers a distance equal to the sum of the lengths. The distance to be covered is independent of the direction in which it is crossing i.e. whether the two objects are crossing each other in opposite directions or if one is overtaking the other i.e. in same direction, the distance to be covered for one to completely cross the other is the sum of the lengths.



In the problems asked, usually a train is crossing either a pole or a man (stationary or walking) or a platform or bridge or another train (obviously moving).

Case 1: Stationary object of negligible length

In this case (e.g lamp posts, pole, a man standing etc) the train should completely cross the object so distance it will cover will be the length of the train.

Case 2: Stationary object of some length

In this case (e.g. bridge, tunnel, platform etc) the last coach of the train should have completely crossed the object so distance it will cover will be length of train + length of the object

Case 3: Moving object of negligible length

In this case (e.g a man walking, a car running etc) the distance the train has to cover to cross the object will be the length of the train. The speed will be the relative speed of the train with respect to the object.

Case 4: Moving object of some length

In this case (e.g. two trains crossing each other) the distance is the total length of the two trains and the speed is the relative speed of one train with respect to another.

Alternately, for all such cases you could use the following with the modifications as noted below:

Time taken to cross = $\dfrac{L_1 + L_2}{S_1 \pm S_2}$

If one of the object is a pole or a man, its length will be 0 (zero)

If one of the object is stationary (e.g. pole, platform, bridge), its speed will be 0 (zero) and the relative speed will just be the speed of the moving object.

E.g. 21: A train running at 72 kmph crosses a telephone booth in 7 sec. What is the length of the train?

Since a telephone booth is a stationary object of negligible length, the distance the train covers is just its own length.

Speed in m/s = $72 \times \dfrac{5}{18}$ = 20 m/s. Thus, distance = 20 × 7 = 140 m.

So length of the train = 140 m

E.g. 22: A train crosses 2 platforms of length 400 m and 600 m in 6s and 8s respectively. What is the length of the train?

Let the length of the train be x.

Then in 6 s, its covers a distance of 400 + x

In 8 s, it covers a distance of 600 + x

But the speed of the train is the same, so $\frac{400+x}{6} = \frac{600+x}{8}$

Cross multiplying $2x = 400$

So we get the value of x as 200. Hence the length of the train is 200m.

E.g. 23: A train consists of 12 coaches, each coach 15 m long. The train crosses another train of length 100 m running in opposite direction in 7 s. Due to some problem, 2 coaches are detached from the first train. In how much time do the trains now cross each other?

Here the length of the first train can be calculated as 12 ×15 = 180 m

Length of the second train = 100m

So total distance that needs to be covered = 180 + 100 = 280m

Since they take 7s to cross each other, Relative speed = 280/7 = 40 m/s

Now, if 2 coaches are detached, there are only 10 coaches so length of train = 10 × 15 = 150 m

So now total distance that needs to be covered = 150 + 100 = 250 m

Since relative speed is 40 m/s, hence time taken to cross = 250/40 = 6.25s

Exercise

23. A train travelling at 100 kmph overtakes a motorbike travelling at 64 kmph in the same direction in 40 seconds. What is the length of the train in meters if initially the motorbike is parallel to the starting point of the train? Assume length of motorbike to be negligible.

    a. 400 m b. 100 m c. 300 m d. 200 m

24. A train travelling at 72 kmph crosses a platform in 30 seconds and a man standing on the platform in 18 seconds. What is the length of the platform in meters?

    a. 360 m b. 240 m c. 120 m d. 200 m

25. Two trains, 200 and 160 meters long take a minute to cross each other while travelling in the same direction and take only 10 seconds when they cross in opposite directions. What are the speeds at which the trains are travelling (in m/s)?

    a. 15, 20 b. 15, 21 c. 18, 21 d. 15, 18

26. Two trains running at 20 m/s and 35 m/s cross the same tunnel in 10s and 12s respectively. What is the difference in the length of the two trains?

    a. 120 m b. 220 m c. 320 m d. 420 m

27. A and B are two stations 390 km apart. A train starts from A at 10:00 am and travels towards B at 65 kmph. Another train starts from B at 11:00 am and travels towards A at 35 kmph. At what time do they meet?

    a. 3:20 pm b. 1:15 pm c. 2:15 pm d. 3:15 pm

28. A goods train leaves a station at a certain time and at a fixed speed. After 6 hrs, an express train leaves the same station and moves in the same direction at a uniform speed of 90 kmph. This train catches up with the goods train in 4 hrs. Find the speed of the goods train.

a. 36 kmph b. 12 kmph c. 18 kmph d. 20 kmph

29. A train can travel 50% faster than a car. Both start from point A at the same time and reach point B 75 kms away from A at the same time. On the way however, the train lost 12.5 minutes while stopping at stations. What is the speed of the car?

a. 100 kmph b. 80 kmph c. 120 kmph d. 60 kmph

Boats and Streams

When a boatman is rowing in still water, say a lake, he would be moving at a speed at which he can row. This speed is called the speed of boat in still water or simply speed of boat. But consider the same boatman in a stream. Because of the current he is either aided (if rowing in the direction of the stream, this is called Downstream) or will be opposed (if rowing against the stream, called Upstream).

If the speed of boatman in still water is B and the speed of the stream is S, we have

Downstream Speed = $B + S$

Upstream Speed = $B - S$

If the speed of boatman is lesser than the speed of the current or stream, the upstream speed will be negative i.e. he is trying to row upstream, but rather than move in that direction, he is taken in opposite direction by the stream. But such situations do not occur in math problems on this topic.

The above relations can be modified for cases when the effective downstream speed, D and upstream speed, U is given and we are supposed to find B and S.

$$B = \frac{D-U}{2},\ S = \frac{D-U}{2}$$

E.g. 24: A boat running downstream covers a distance of 16 km in 2 hrs while for covering the same distance upstream, it takes 4 hrs. What is the speed of the boat in still water?

Speed Upstream = 16/2 = 8 kmph

Speed Downstream = 16/4 = 4 kmph

From the above formula, speed of boat in still water = $\frac{400+x}{6} = \frac{600+x}{8}$ = 6 kmph

E.g. 25: A man can row at a speed of 7.5 kmph in still water. If in a river running at 1.5 kmph, it takes him 50 min to row to a place and back, how far off is the place?

Here Speed Downstream = 7.5 + 1.5 = 9 kmph

Speed Upstream = 7.5 – 1.5 = 6 kmph

Let the required distance be x. Total time taken by him is 5/6 hr when he rowed once upstream a distance of x and once downstream a distance of x.

So $\frac{x}{9} + \frac{x}{6} = \frac{5}{6}$

Solving we get, $x = 3$

Hence the place is 3 km away.

Exercise

30. The speed of a boat in still water is 10 m/s and the speed of the stream is 6 m/s. If the boat is moving downstream and then upstream, what is the ratio of the time taken to cover a particular stretch of distance in each direction?

    a. 1/4 b. 1/2 c. 2/3 d. 1/3

31. A boat takes a total of 6 hours to row 8 kms downstream and to return back to the starting point. If speed of the boat is 3 kmph, for how much time was the boat moving downstream and for how much time upstream?

    a. 2 hrs, 4 hrs b. 5 hrs, 1 hrs c. 3 hrs, 3 hrs d. 4.2 hrs, 1.8 hrs

32. A man rows for 3 hrs downstream and then for 3 hrs upstream. In this whole process he covers a distance of 12 kms. If the speed of the stream is 1 kmph, find the speed of the boat in still water.

    a. 2 kmph b. 2.5 kmph c. 3 kmph d. Cannot be determined

33. A boat travels from point A to point B upstream and returns from point B to point A downstream. If the round trip takes the boat 5 hours and the distance between point A and point B is 120 kms and the speed of the stream is 10 kmph, how long did the upstream journey take?

    a. 3.5 hrs b. 3 hrs c. 2 hrs d. 1.5 hrs

34. The speed of a motor boat itself is 20 km/h and the rate of flow of the river is 4 km/h. Going downstream, the boat went 120 km. What distance will the boat cover during the same time going upstream?

    a. 100 km b. 120 km c. 80 km d. 90 km

35. A man can row 50 km upstream and 72 km downstream in 9 hours. He can also row 70 km upstream and 90 km downstream in 12 hours. Find the rate of current.

    a. 14 kmph b. 12 kmph c. 4 kmph d. 2 kmph

Time and Work

In questions on this topic, we usually have two or more people working simultaneously on a work. With two or more working together, the only aspect to watch out for is that the work done by each is additive (and NOT the days). Thus, if A can finish a work in 3 days and B can finish a work in 4 days, the number of days taken when both work simultaneously will not be 3 + 4 = days. This is very foolish since when both are working together, they will obviously finish the work in lesser number of days taken by either of them individually. So remember that work done needs to be added and days taken should NEVER be added.

If a person does a work in a days, then every day he does $\frac{1}{a}^{th}$ of the work.

Conversely, if in one day, $\frac{1}{x}^{th}$ of a work is done, the entire work can be done in x days.

Let's learn the two approaches to solve such questions through an example:

Aatish can do a work in 20 days. Azad can do the same work in 15 days. In how many days can both of them together complete the work?

Approach 1: Adding Per day's Work:

Since A does the work in 20 days, everyday he does $\frac{1}{20}^{th}$ of the work.

Since B does the work in 15 days, everyday he does $\frac{1}{15}^{th}$ of the work.

If both work together, everyday they complete $\frac{1}{20}+\frac{1}{15}=\frac{7}{60}^{th}$ of the work.

So number of day required to finish the work = 60/7 = approximately 8.5 days

Approach 2: LCM Approach.

In this approach we assume the work as a numeric quantity, usually the LCM of the days taken individually (this will avoid working with any fractions). After assuming the work, we find the individual rate of working.

Assuming work as LCM of 15 and 20 i.e. 60 units …

Aatish does 60 units in 20 day i.e. his rate of working is 3 units per day.

Azad does 60 units in 15 day i.e. his rate of working is 4 units per day.

When both work simultaneously, the effective rate is 3 + 4 = 7 units per day.

Since total work is 60 units, day, taken will be 60/7 i.e. approximately 8.5 days.

E.g. 1: Parul is twice as efficient as Parag. If both working together complete a work in 10 days, how many days would each have taken individually?

Approach 1: Adding Per day's Work:

Since Parul is twice as efficient as Parag, if Parag takes $2x$ days to complete a work, Parul will take x days to complete the same work.

Thus, work done by both of them in 1 day will be $\frac{1}{2x}^{th}+\frac{1}{x}^{th}$ i.e. $\frac{3}{2x}^{th}$ of the total work.

Since both work together they finish a work in 10 days, work done in 1 day will be $\frac{1}{10}^{th}$ of the work.

Equating the two, $\frac{3}{2x} = \frac{1}{10} \Rightarrow x = 15$

So Parag would have taken 30 days to complete the work independently and Parul would have taken 15 days.

Approach 2: LCM Approach.

Assuming work as 10 units and since it is done in 10 days, the rate of doing work is 1 unit per day.

Now if Parag does x unit's of work in 1 day, Parul being twice as efficient will do $2x$ units of work in 1 day.

Thus, $x + 2x = 1$ i.e. $x = \frac{1}{3}$

Parag does 1/3 units per day and to do 10 units will require $\frac{10}{1/3} = 30$ days.

Parul does 2/3 units per day and to do 10 units will require $\frac{10}{2/3} = 15$ days.

E.g. 2: Ishan can do a certain job in 18 days. He works at it for 10 days and then Bharat alone completes it in 4 days. In how many days will Ishan and Bharat working together, finish the work?

Approach 1: Adding Per day's Work:

Since Ishan does the work in 18 days, it means he does $\frac{1}{18}^{th}$ of the work everyday.

So in 10 days, he does $\frac{10}{18}^{th}$ of the work. Remaining work is $\frac{8}{18}^{th}$

Since Bharat completes the rest of the work in 4 days, everyday he does $\frac{8/18}{4} = \frac{1}{9}^{th}$ of the work.

So if they both work together, everyday they do $\frac{1}{18} + \frac{1}{9} = \frac{1}{6}^{th}$ of the work.

Hence working together, they will finish the work in 6 days.

Approach 2: LCM Approach.

Assume the work as 18 units. Since Ishan does the work in 18 days, his rate of working is 1 unit per day. When he works for 10 days, he will complete 10 units of work.

The remaining 8 units of work is done by Bharat in 4 days i.e. his rate of working is 2 units per day.

Together their rate is 1 + 2 = 3 units per day and to complete 18 units of work, they will require 18/3 = 6 days.

E.g. 3: A man and two boys can do a piece of work in 10 days. Two men and a boy can do the same work in 8 days. In how many days can a man and a boy together complete the work?

Approach 1: Adding Per day's Work:

A man and two boys complete $\frac{1}{10}^{th}$ of the work everyday.

Two men and a boy complete $\frac{1}{8}^{th}$ of the work everyday.

Then, three men and three boys together will complete $\frac{1}{8}+\frac{1}{10}=\frac{9}{40}^{th}$ of the work everyday.

This means that one boy and one man will together do $\frac{9}{40}\times\frac{1}{3}=\frac{3}{40}^{th}$ of the work everyday.

So they need 40/3 = 13.33 days to finish the work.

Approach 2: LCM Approach.

Let the work be LCM of 10 and 8 i.e. 40 units.

Further let each man do m units per day and each boy do b units per day.

Since a man and two boys can do 40 units in 10 days i.e. 4 units per day, hence $m + 2b = 4$

Since two men and a boy can do 40 units in 8 days i.e. 5 units per day, hence $2m + b = 5$

Adding the two $3(m + b) = 9$ i.e. $m + b = 3$

Thus, 1 man and 1 boy can do 3 units per day and to do 40 units, they would take 40/3 days.

E.g. 4: One man can build a bridge in 100 days. One woman can build the same bridge in 120 days. One child can destroy the bridge completely in 200 days. If two men, three women and two children are working simultaneously on the bridge, in how many days will it be complete?

Approach 1: Adding Per day's Work:

One man does $\frac{1}{100}^{th}$ of the work everyday; One woman does $\frac{1}{120}^{th}$ of the work everyday; One child destroys $\frac{1}{200}^{th}$ of the work everyday.

Two men do $\frac{1}{50}^{th}$ of the work everyday; three women do $\frac{1}{40}^{th}$ of the work everyday and two children destroy $\frac{1}{100}^{th}$ of the work everyday.

Hence in all, the work done per day = $\frac{1}{50}+\frac{1}{40}-\frac{1}{100}=\frac{4+5-2}{200}=\frac{7}{200}^{th}$ of the work.

Hence no of days needed to finish the work = 200/7 = 28.56 days

Approach 2: LCM Approach.

Assume the work to be done as LCM of 100, 120 and 200 i.e. 600 units.

Since 1 man does 600 units in 100 days, one man's rate of working is 6 unit's per day.

Since 1 woman does 600 units in 120 days, one woman's rate of working is 5 unit's per day.

Since 1 child destroys 600 units in 200 days, one child's rate of working is 2 unit's per day.

When 2 men, 3 women and 2 children work together, the net rate of working will be 2 × 6 + 3 × 5 – 2 × 3 = 12 + 15 – 6 = 21 units per day.

Thus, days taken to do 600 units will be 600/21 i.e. 200/7 days.

E.g. 5: Pipe A running alone can fill a cistern in 10 hours. Pipe B running alone can fill a cistern in 12 hrs. In how many hours will both running together fill the cistern?

Every hour Pipe A fills $\frac{1}{10}^{th}$ of the cistern.

Every hour Pipe B fills $\frac{1}{12}^{th}$ of the cistern.

This means that every hour, when both are running together they will fill $\frac{1}{10}+\frac{1}{12}=\frac{11}{60}^{th}$ of the tank.

Hence the two pipes running together need 60/11 = 5.45 hours

Exercise

1. Six men can do a piece of work in 15 days. How many men are needed to complete the work in 12 days if they are half as efficient as the six men?

   a. 8 b. 12 c. 15 d. 10

2. Two men and four women do a job in 3 days and three men and one women do the same job in 4 days. How long will one man and two women take to finish the same work?

   a. 5 b. 6 c. 8 d. 10

3. A and B can do a work in 8 days; B and C can do it in 12 days; A, B and C together can finish it in 6 days. In how many days will A and C together do it?

   a. 7 b. 8 c. 6 d. 5

4. A takes twice as much time as B or thrice as much time as C to finish a piece of work. Working together they can finish the work in 4 days. In how many days can B alone do the work?

   a. 12 b. 10 c. 11 d. 14

5. A can do a piece of work in 14 days while B can do it in 21 days. They begin together but 3 days before the completion of the work, A leaves off. What is the total number of days required to complete the work?

   a. 10 b. 10.2 c. 10.5 d. 11

6. A can do a work in 3 days while B can do the same work in 2 days. Both of them finish the work together and get Rs. 200. What is the share of A?

   a. Rs. 120 b. Rs. 100 c. Rs. 80 d. Rs. 60

7. Twenty women can do a work in 16 days. Sixteen men can complete the same work in fifteen days. What is the ratio between the capacity of a man and a woman?

   a. 3 : 4 b. 2 : 3 c. 2 : 1 d. 4 : 3

8. 12 men complete a work in 9 days. After they have worked for 6 days, 6 more men join them. How many days will they take to complete the remaining work?

   a. 3 b. 2 c. 1 d. 1.5

9. A man, a woman and a boy can complete a job in 3, 4 and 12 days respectively. How many boys must assist 1 man and 1 woman to complete the job in $\frac{1}{4}^{th}$ of a day?

   a. 5 b. 41 c. 24 d. 36

10. Four men and six women can complete a work in 8 days, while three men and seven women can complete it in 10 days. In how many days will 10 women complete it?

    a. 40 b. 20 c. 10 d. 30

11. A cistern normally fills up in 10 hours. However it takes 12 hours when there is a leak in its bottom. If the cistern is full, in what time shall the leak empty it?

    a. 20 hrs b. 60 hrs c. 30 hrs d. 15 hrs

12. Two taps running simultaneously fill a tank. The first tap could have filled it in 8 hrs by itself, the second tap could have filled it in 24 hrs by itself. But due to a leak in its bottom, there was a delay in filling it up by 1 hr. Find the time in which the leak will empty a full tank.

    a. 35 hrs b. 36 hrs c. 30 hrs d. 42 hrs

Assignment: Time Speed & Distance & Work

1. Ramu has a habit of converting values in different units. He wants to convert a speed of 3 miles per hour to its meters per second equivalent. What will the equivalent speed be? (assume 1 mile = 1.6 km)

   (a) 1.33 m/s (b) 0.833 m/s (c) 10.8 m/s (d) 1.5 m/s

2. Akash travels from his office to his home at a speed of 40 kmph and take 90 mins for the journey. But today he wants to finish the journey in 60 mins. At what speed should he travel today?

   (a) 70 kmph (b) 45 kmph (c) 50 kmph (d) 60 kmph

3. A person makes a journey of 200 kms in two legs. In first leg he travels for 7 hrs at a speed of 20 kmph. And the second leg of journey takes 2 hours. At what speed does he travel in the second leg of the journey?

   (a) 40 kmph (b) 60 kmph (c) 30 kmph (d) 25 kmph

4. Ratio of speeds of Seeta and Geeta is 3 : 2. If Geeta takes 27 minutes to travel a certain distance, how much time will Seeta take to travel double that distance?

   (a) 40 mins (b) 36 mins (c) 60 mins (d) 54 mins

5. Three sprinters run a 100 meters race. If their speeds are in the ratio 3 : 4 : 5, find the ratio of the time taken by them to cover the distance.

   (a) 4 : 3 : 2 (b) 5 : 4 : 3 (c) 20 : 15 : 12 (d) 16 : 15 : 12

6. Travelling at $3/5^{th}$ of my usual speed, I take 14 minutes more than usual to reach office from home. What is the usual time taken for the journey?

   (a) 21 mins (b) 28 mins (c) 35 mins (d) 42 mins

7. Hawaldar Rampyare sees a thief 1200 mtr ahead of him. The thief also sees Rampyare simultaneously and starts running in the opposite direction at the speed of 20 mtrs/sec. At what speed should the hawaldar run to catch the thief in exactly 3 mins?

   (a) 50 mtrs/sec (b) 40 mtrs/sec (c) 60 mtrs/sec (d) 35 mtrs/sec

8. Two people A and B are 700 mts away and they start running towards each other at speeds of 2.5 mts/sec and 4.5 mts/sec respectively. How far, from the point where A started, will they meet?

   (a) 350 mts (b) 450 mts (c) 250 mts (d) 500 mts

9. Two trains A and B of lengths 250 m and 350 m, travelling at speeds of 50 m/s and 108 km/h respectively, are traveling in the same direction. How much time will train A take to completely cross train B?

   (a) 10 mins (b) 40 secs (c) 1/2 min (d) 4 mins

10. A train takes 10 secs to cross a pole and 20 seconds to cross a platform. If the length of the platform is 100 mts, what is the length of the train?

    (a) 50 mts (b) 75 mts (c) 100 mts (d) 125 mts

11. A man rows 24 km upstream and 12 km downstream in four hours each. What is the speed of boat and that of the stream?

    (a) 4 m/s, 2 m/s (b) 6 m/s, 3 m/s (c) 4.5 m/s, 1.5 m/s (d) 1.5 m/s, 4.5 m/s

12. In a stream are two boats, 300 meters apart from each other. James Bond in boat A wants to catch Dr. Chang in boat B. Speeds of boats A and B are 50 mtrs/sec and 30 mtrs/sec. How long will it take for James Bond to catch Dr. Chang?

    (a) 15 sec (b) 18 sec

    (c) 16 sec (d) Depends on speed of stream.

13. Shyam breaks his journey in two equal time intervals, the first of which he travels at 30 kmph and the second at 60 kmph. What is his average speed?

    (a) 25 kmph (b) 60 kmph (c) 45 kmph (d) 40 kmph

14. If I travel from Lucknow to Kanpur at a speed of 30 kmph and on the return journey, I travel at a speed of 60 kmph, what is my average speed for the entire round trip?

    (a) 25 kmph (b) 60 kmph (c) 45 kmph (d) 40 kmph

15. A can do a work in 20 days working alone and B can complete the same job working alone in 30 days. A and B work together on the work for 6 days and then B leaves and the remaining work is done by A alone. How many total days are needed to finish the work?

    (a) 15 (b) 16 (c) 18 (d) 21

16. Pipe A can empty half a tank in 30 mins and the same tank can be filled completely by pipes B and C working together in12 mins. If all three pipes, A, B and C, are opened together, how long would it take to fill the entire tank?

    (a) 15 mins (b) 25 mins (c) 30 mins (d) Cannot be determined

17. A tank can be filled by the inlet pipe in 6 hours. But today it took 7 hours to fill the tank because of a leak at the bottom of the tank. The leak alone will empty the filled tank in how many hours?

    (a) 36 hours (b) 42 hours (c) 49 hours (d) 50 hours

18. The work done by a man in 4 days is equal to the work done by a boy in 6 days. If it takes 2 men and 3 boys to do a certain work in 12 days, how many days are required to do the same work by 4 men and 3 boys?

    (a) 10 (b) 8 (c) 9 (d) 6

19. A is thrice as fast a work-man as B. If a task can be finished in 1 day if both A and B work simultaneously, how many days will B take to finish the task working alone?

    (a) 2 days (b) 3 days (c) 4 days (d) 6 days

20. If 1 man or 2 women can finish a work in 10 days, in how many days will 1 man and 2 women finish the work?

    (a) 8 (b) 6 (c) 5 (d) 4

Algebra

Algebra is distinct from Arithmetic because of the presence of a variable. A variable, as the name suggests, is a quantity that can assume different values. It is typically denoted by any alphabet, most commonly used ones are x, y, z, a, b, m, n, p, q, r, etc.

One does not really need to get into a lot of theoretical knowledge about terminology of Algebra like polynomials, degree, roots, etc and one can directly solve questions based on a very elementary awareness also. So let's directly deal with the types of questions.

Linear Equations in one variable

Linear equations are those where the highest index of the variable is just 1 i.e. $2x + 5 = 10$ is a linear equation but $x^2 - 2x + 4 = 0$ is not a linear equation.

Solving of linear equations in one variable can easily be done by transposing all terms involving the variable on one side of the equality and the constant terms on the other side.

E.g. 1: Solve for x: $5x - 8 = 3x$

Transposing, $5x - 3x = 8$ i.e. $2x = 8$ i.e. $x = 4$.

E.g. 2: Solve for x: $\frac{2}{5x} - \frac{1}{15} = \frac{5}{2x}$

Transposing, $\frac{2}{5x} - \frac{5}{2x} = \frac{1}{15}$

$$\frac{4-25}{10x} = \frac{1}{15} \Rightarrow \frac{-21}{10x} = \frac{1}{15}$$

Cross-multiplying, $10x = -21 \times 15 \Rightarrow x = -\frac{63}{2}$

E.g. 3: Solve for x: $\frac{3x+5}{3-2x} = \frac{5}{3}$

Cross-multiplying, $9x + 15 = 15 - 10x$

Transposing, $9x + 10x = 15 - 15$, i.e. $19x = 0$ i.e. $x = 0$.

But obviously one does not get as easy questions as these in the exam. In the exam you would have to frame the equations yourself before solving them. A worded problem will be given that has to be translated into an equation. To be able to do so, go through the following data and see how different pieces of given data suggest which quantity to assume as a variable and how to form the expression/equation:

Given data	Quantity assumed as variable	Expression/Equation formed
When difference of two numbers is given		
First number is 3 more than the second number.	Let second number be x	$\therefore$, the first number = x +3
When the sum of two number is given		
The sum of two number is 40	Let one number be x	$\therefore$ second number = 40 – x
When one number is given as a multiple of other		
First number is 3 times the second number	Let second number be x	$\therefore$ first number = $3x$
When the ratio of two numbers is given		
The ratio of two numbers is 3:5	Let the first number be $3x$	$\therefore$ the second number = $5x$
Consecutive Numbers		
Three consecutive numbers	Let the first number be x	Then the numbers are x, $x + 1$, $x + 2$
Consecutive even or odd numbers		
Four consecutive even (or odd) numbers	Let the first number be x	Then the numbers are x, $x + 2$, $x + 4$, $x + 6$
Data regarding ages		
5 years hence the age of a father will be…	Let the present age of father be x years	$\therefore$ 5 years hence age of father will be $x + 5$
Six years ago Mohan was three times as old as Shyam.	Let the present ages of Mohan and Shyam be m and s respectively	6 years ago their age would have been $m – 6$ and $s – 6$ resp Based on the relation given $(m – 6) = 3 \times (s – 6)$.
Data regarding dividing an amount among persons		
Rs. 1000 is divided among certain number of men equally.	Let the number of men be n.	$\therefore$ each man's share = $\frac{1000}{n}$

E.g. 4: Two numbers are in the ratio 4 : 7. When each of the number is increased by 5, the ratio becomes 3 : 5. Find the numbers.

Let the numbers be $4x$ and $7x$.

Thus, $\frac{4x+5}{7x+5} = \frac{3}{5}$

$20x + 25 = 21x + 15$

$x = 10$ and the numbers are 40 and 70.

E.g. 5: Three times the first of three consecutive odd integers is 3 more than twice the third number. Find the middle number.

Let the numbers be x, $(x + 2)$ and $(x + 4)$

We have $3x = 2(x + 4) + 3$

$3x = 2x + 8 + 3$

$x = 11$

Thus the middle number is 13.

Given data	Quantity assumed as variable	Expression/Equation formed
Data based on rate and expenditure		
Finding the expression for the number of articles bought in Rs. 20	Let the price per article be Rs. p	$\therefore$ number of articles bought in Rs. 20 $= \frac{20}{p}$
When the price per article is increased by Rs. 5, I can purchase 8 articles less in Rs. 20.	Let the original price per article be Rs. p.	The new price per article = Rs. $p + 5$ Original quantity bought = $\frac{20}{p}$ Quantity bought now = $\frac{20}{p+5}$ As per the relation, diff in quantity is 8, so $\frac{20}{p} - \frac{20}{p+5} = 8$
Eight pens and 6 pencils cost Rs. 100	Let the price of one pen be Rs. x and of one pencil be Rs. y.	$\therefore\ 8x + 6y = 100$
Relation between digits of a two digit number		
The unit's digit of a two digit number is 4 more than the ten's digit.	Let the two digit number be xy	$y = x + 4$
Value of a two digit number		
When the digits of a two digit number are interchanged, the number increases by 45	Let the two digit number be xy. On interchanging the digits, we get the number yx.	The value of xy is $(10x + y)$ The value of yx is $(10y + x)$ Since the value of interchanged number is 45 more, $(10y + x) - (10x + y) = 45$

The above table captures the most common data that come in problems. The methods citied above are not exhaustive and nor are they the only way to represent. If one chooses some other quantity as the variable, one would get a different expression. Whatever is given is for you to get started with problems. Once you have enough practice, you would be more confident in choosing any quantity as the variable and then forming the needed equation.

E.g. 6: A two digit number exceed the sum of the digits of that number by 18. If the digit in the unit's place is double the digit in the ten's place, what is the number?

Since the digit in the unit's place is double the digit in the ten's place, lets assume the digit in tens place to be x .Then the digit in the unit's place is $2x$.

The value of the number is $10x + 2x = 12x$

Since the value of the number exceeds the sum of digit by 18,

$12x = x + 2x + 18$

$9x = 18$ i.e. $x = 2$.

Thus the digit in ten's place is 2 and the digit in unit's place is 4 and the number is 24.

E.g. 7: The sum of the present ages of a father and son is 60 years. Six years ago, father's age was five times the age of the son. Find the present age of the son.

Let the present age of the son be x. Hence the age of the father will be $60 - x$.

Six years ago their ages would have been $(x - 6)$ and $(60 - x - 6)$ i.e. $(54 - x)$

Thus by the relation given, $(54 - x) = 5 \times (x - 6)$

$54 - x = 5x - 30$

$6x = 84$ i.e. $x = 14$.

Thus the son's present age is 14 years.

E.g. 8: Certain amount was divided among 24 people. If there would have been 6 less men, each would have received Rs. 2 more. Find the amount distributed.

Let the amount distributed be x, then by the relation given

$$\frac{x}{18} - \frac{x}{24} = 2$$

$$\frac{4x - 3x}{6 \times 3 \times 4} = 2 \Rightarrow x = 144$$

Exercise

1. A number added to two-seventh of its value is equal to 72. Find the number.

   a. 42 b. 56 c. 49 d. 35

2. The sum of four consecutive odd numbers is 80. Find the average of the four numbers.

   a. 20 b. 10 c. 15 d. 18

3. What is the sum of two consecutive multiples of 3, the difference of whose squares is 81?

   a. 15 b. 24 c. 27 d. 51

4. The sum of two numbers is 50. The fraction obtained by dividing the larger number by the smaller number is $\frac{3}{2}$. Find the numbers.

   a. 20, 30 b. 15, 35 c. 24, 26 d. None of these

5. Six years ago, Sushil's age was thrice that of Snehal's age. Six years later, Sushil's age will be $\frac{5}{3}$ times the age of Snehal. What is the present age of Snehal?

   a. 16 b. 14 c. 12 d. 18

6. The sum of the digits of a two digit number is 9 less than the number. Find the digit in the ten's place of the number.

   a. 4 b. 3 c. 2 d. 1

Linear Simultaneous Equations

Again as the name suggests, these are linear equations. The difference from the earlier topic is that each equation has two variables. Thus the equation will look like $4x + 3y = 50$. The solution to this equation is a pair of value of x and y that satisfy the equation. Since there are two variables, and only one equation, we can have infinite solutions to the equation. Take any value for one variable, and substituting that value in the equation, we will get a linear equation in one variable, solving which we can find the value of the other variable. E.g. In the equation $4x + 3y = 50$, if $x = 0$, then we have $3y = 50 \Rightarrow y = \frac{50}{3}$. Thus the pair $x = 0$ and $y = \frac{50}{3}$ satisfy the given equation. We could also have assumed $x = 1$ instead of 0. In this case we would have $4 + 3y = 50 \Rightarrow 3y = 46 \Rightarrow y = \frac{46}{3}$. Thus the pair, $x = 1$ and $y = \frac{46}{3}$ would also satisfy the equation. We could have assumed any value for x and found a corresponding value of y that could satisfy the equation. So, we would have infinite solution to the equation.

However consider the set of equations:

$4x + 3y = 50$ and $5x - 2y = 10$.

There would be infinite solutions to each of the equations considered independently. But considered simultaneously i.e. a solution that satisfies both the equations (a pair of values of x and y that satisfy both the equation), it is quite possible that there is just one solution.

Solving for a simultaneous solution to a system of two (or more) equations in two (or more) variables is called Simultaneous Equations.

Following are the steps involved in solving a system of two simultaneous linear equations:

Step 1: If needed multiply the equations by constants so as to make the coefficient of any one of the variable, numerically the same in both the equation. If the coefficient of any one of the variable is already numerically the same, there is no need to perform this step.

Step 2: Either add the two equations or subtract the two equations so that the variable with numerically equal coefficient gets eliminated. This step will result in a singe linear equation in one variable.

Step 3: Solve the linear equation in one variable obtained in step 3.

Step 4: Substitute the value of the variable just found in either of the equation to find the value of the other variable. The two values found for the two variables is a simultaneous solution to the system to two equations

E.g. 9: To explain the above steps, lets solve the set of equations

$4x + 3y = 50$ ....(i)

and $5x - 2y = 10$. ....(ii)

Step 1: Let's make the co-efficient of y to be equal since they are smaller numbers.

(i) × 2 $8x + 6y = 100$ ....(iii)

(ii) × 3 $15x - 6y = 30$ ....(iv)

Step 2: Since the coefficients of y are of opposite signs, adding them will eliminate them.

(iii) + (iv) $23x = 130$

Step 3: Solving $23x = 130$, we get $x = \frac{130}{23}$

Step 4: Substituting the value of x just found in equation (i),

$$4 \times \frac{130}{23} + 3y = 50$$

$$\Rightarrow \frac{520}{23} + 3y = 50 \Rightarrow 3y = 50 - \frac{520}{23}$$

$$\Rightarrow 3y = \frac{1150 - 520}{23} = \frac{630}{23}$$

$$\Rightarrow y = \frac{210}{23}$$

Thus the solution that satisfies both the equations is $x = \frac{130}{23}$ and $y = \frac{210}{23}$

No Solution, Infinite Solution and Unique Solution

Not always does a system of two linear equations in two variables have a simultaneous solution.

It is quite possible that there is no common solution among the infinite solutions of each of the equations. In this case there would be no solution that would satisfy both the equations simultaneously.

Condition for No Simultaneous Solution

Consider the two equations in x and y as $Ax + By = C$ and $Px + Qy = R$

If $\frac{A}{P} = \frac{B}{Q} \neq \frac{C}{R}$, the set of equations will not have any common solution and the two equations are said to be Inconsistent.

E.g. 10: 8 pens and 6 pencils together cost Rs. 100 whereas 4 pens and 3 pencils together cost Rs. 60. Find the cost of a pencil and a pen.

The two data given boil down to

$8x + 6y = 100$ and $4x + 3y = 60$

Since $\frac{8}{4} = \frac{6}{3} \neq \frac{100}{60}$, the set of equation will not have any common solution and the data given is
inconsistent. (Obviously, if 8 pens and 6 pencils costs Rs. 100, how much should half the material i.e. 4 pens and 3 pencils cost? Rs. 50, right? The data mentions Rs. 60, hence the inconsistency.)

Condition for Inifinite solutions

This is the case where there are more than one solution that simultaneously satisfy both the equations. In such a case, if there are more than one solution, there would be infinite solutions that would satisfy both the equations simultaneously. Infact, each solution of the infinitely many solutions of the first equation, would also satisfy the second equation.

Condition for Infinite Simultaneous Solutions:

Consider the two equations in x and y as $Ax + By = C$ and $Px + Qy = R$

If $\frac{A}{P} = \frac{B}{Q} = \frac{C}{R}$, the set of equations will not have any common solution and the two equations are said to be Dependent.

E.g. 11: If 15 apples and 21 mangoes together cost Rs. 300 and 10 apples and 14 mangoes together cost Rs. 200, find the cost of 1 apple.

The equations are $15x + 21y = 300$ and $10x + 14y = 200$.

Since, $\frac{15}{10} = \frac{21}{14} = \frac{300}{200}$, the set of equation has infinite solution and thus no unique value for the price of apple can be found.

Unique Solution

The given equations, say

$Ax + By = C$ and

$Px + Qy = R$

would have a unique simultaneous solution only when $\frac{A}{P} \neq \frac{B}{Q}$

Thus the above can be captured into the following snapshot:

$Ax + By = C,$
$Px + Qy = R$

- Inconsistent Equations
  No solution
  $\frac{A}{P} = \frac{B}{Q} \neq \frac{C}{R}$
- Dependent Equations
  Infinite solution
  $\frac{A}{P} = \frac{B}{Q} = \frac{C}{R}$
- Unique solution
  $\frac{A}{P} \neq \frac{B}{Q}$

Thus before beginning to solve any simultaneous equation, it's a good idea to have a look at the ratios $\frac{A}{P}, \frac{B}{Q}, \frac{C}{R}$ to first identify if you can get a unique solution.

Exercise

7. 33 pencils and 47 erasers together cost Rs. 299, while 47 pencils and 33 erasers cost Rs. 341. Find the cost of purchasing one pencil and one pen.

   a. 8 b. 10 c. 12 d. 14

8. Four computers and three printers together cost Rs. 94,000, while six computers and eight printers cost Rs. 1,62,000. Find the cost of one computer.

   a. 20,000 b. 19,000 c. 18,000 d. 21,000

9. In a two digit number, the digit in unit's place is 1 more than twice the digit in ten's place. If the digit in the unit's and ten's place is interchanged, we get a number which is 1 less than twice the original number. Find the number.

   a. 49 b. 25 c. 37 d. 73

10 There are two numbers such that the sum of twice the first and thrice the second is 39 whereas the sum of thrice the first and twice the second is 36. Find the larger number.

   a. 6 b. 9 c. 12 d. 8

11. The difference between a two digit number and the number obtained by interchanging the positions of the digits is 36. What is the difference between the two digits of the number?

   a. 4 b. 6 c. 12 d. 5

12. The number obtained on interchanging the positions of the digits of a two digit number is 18 more than the original number. If the sum of the digits is 8, find the number.

   a. 35 b. 53 c. 27 d. 72

13. The difference between two number is 642. When the larger number is divided by the smaller number, the quotient is 8 and the remainder is 19. Find the larger number.

   a. 731 b. 273 c. 137 d. 89

14. Solve for p and q: $3(2p + q) = 7pq$ and $3(p + 3q) = 11pq$.

   a. 3/2, 1 b. 1, 4/3 c. 1, 3/2 d. 4/3, 1

15. In a given fraction, if the numerator is increased by 3 and the denominator is reduced by 3, the resulting fraction is 5/6. But if the numerator is increased by 4 and denominator is decreased by 4, the resulting fraction is 6/5. Find the original fraction.

   a. 3/5 b. 5/9 c. 2/9 d. 1/7

16. For what value of k would the system of equation have infinite solutions: $12x - 20y = 32$ and $15x - 25y = k$.

   a. 40 b. 44 c. 48 d. 56

Quadratic Equations

A quadratic equation is one in which the highest index of the variable is 2.

The general form of the quadratic equation is $ax^2 + bx + c = 0$.

Please note that a is the coefficient of x^2, b is the coefficient of x and c is the constant term. DO NOT memorize them as the first coefficient is a, second is b and third is c because the order in which the equation is written can be changed. Also the coefficient has to be taken along with their signs. E.g. in the quadratic equation $4x - 10 + 3x^2 = 0$, $a = 3$ (and not 4), b is 4 and c is -10 (and not 10).

There are two ways to solve a quadratic equation, viz. method of factorization or use of formula.

Factorising a quadratic

Consider the above mentioned quadratic and the process of finding its root,

$x^2 - 5x - 6 = 0$

$x^2 - 6x + x - 6 = 0$

$x(x - 6) + 1(x - 6) = 0$

$(x - 6) \times (x + 1) = 0$

$x = 6$ or -1

You must surely have learnt this process ages ago. So, let's refresh it now. The term in x has to be broken into two terms, not any random terms but into two specific terms as explained below.

The method explained below helps you to write the given quadratic expression directly as a product of two factors and thus helps you save valuable time. So read it carefully. If you understand this well enough, you can directly find the roots of a quadratic equation without doing any pencil work.

We know that any quadratic expression can be factorised into two terms i.e.

$x^2 - 5x - 6 = (x + p) \times (x + q)$

All we need to know are the values of p and q. Expanding the RHS, we have

$x^2 - 5x - 6 = x^2 + (p + q)x + pq$

Thus the two numbers that we are searching for, p and q, are such that their sum is -5 and product is -6.

Always start with finding the two numbers that satisfy the product. Two numbers that multiply to -6 are $(-3, 2)$ or $(3, -2)$ or $(-6, 1)$ or $(6, -1)$. The sum of which of these pairs is -5? Obviously the required values of p and q are -6 and 1. Thus, we can directly write the factorised form as

$x^2 - 5x - 6 = (x - 6) \times (x + 1)$

After this finding the roots is just a nominal step ahead.

Examples:

1. $x^2 + 3x - 10 = 0$

   We need two numbers such that the product is -10 and sum is 3. The product, -10 can be formed by $(\pm 10, \mp 1)$ and $(\pm 5, \mp 2)$. Only $(5, -2)$ is a pair that adds up to 3. Thus,

   $x^2 + 3x - 10 = (x + 5) \times (x - 2) = 0$ and the roots are -5 and 2

2. $x^2 - 7x + 12 = 0$

   We need two numbers such that the product is 12 and sum is -7. The product, 12 can be formed by $(\pm 3, \pm 4)$ and the sum of -3 and -4 is -7 . Thus,

   $x^2 - 7x + 12 = (x - 3) \times (x - 4) = 0$ and the roots are 3 and 4

3. $x^2 - 10x + 24 = 0$

We need two numbers such that the product is 24 and sum is –10. The product, 24 can be formed by (±6, ±4) or (±12, ±2) or (±8, ±3). Which pair of these have a sum of –10? Knowing that both numbers have to be negative, its not difficult to find the pair as –6 and –4. Thus,

$x^2 - 10x + 24 = (x - 6) \times (x - 4) = 0$ and the roots are 6 and 4.

Practice

Factorise the following (Answers at end of chapter)

1. $x^2 + 10x + 9 = (x \qquad) \times (x \qquad)$
2. $x^2 + 7x + 12 = (x \qquad) \times (x \qquad)$
3. $x^2 - 2x - 8 = (x \qquad) \times (x \qquad)$
4. $x^2 - 6x - 7 = (x \qquad) \times (x \qquad)$
5. $x^2 + 3x - 4 = (x \qquad) \times (x \qquad)$
6. $x^2 + 4x - 12 = (x \qquad) \times (x \qquad)$
7. $x^2 - 8x + 15 = (x \qquad) \times (x \qquad)$
8. $x^2 + 9x + 18 = (x \qquad) \times (x \qquad)$
9. $x^2 - 10x - 24 = (x \qquad) \times (x \qquad)$
10. $x^2 + 23x + 132 = (x \qquad) \times (x \qquad)$

The difficulty in the above comes when the coefficient of x^2 is not 1 as in the following example:

$4x^2 + 9x + 5 = 0$

Here we should NOT look out for two numbers whose product is 5 and sum is 9. Instead we should look out for two numbers such that whose product is 5 × 4 i.e. 20 and the sum is 9. The numbers are 5 and 4. Now the factors are NOT going to be $(x + 5)$ and $(x + 4)$ but are $\left(x+\frac{5}{4}\right)$ and $\left(x+\frac{4}{4}\right)$. Thus the factorising will be as follows :

$4x^2 + 9x + 5 = 4 \times \left(x+\frac{5}{4}\right) \times \left(x+\frac{4}{4}\right)$ and the roots are $-\frac{5}{4}$ and -1.

Examples:

1. $2x^2 - 13x + 15$

Looking for two numbers whose product is 15 × 2 i.e. 30 and sum is –13, we zero down on –10 and –3. Thus,

$2x^2 - 13x + 15 = 2 \times \left(x+\frac{10}{2}\right)\times\left(x-\frac{3}{2}\right)$ and the roots are –5 and $\frac{3}{2}$

2. $6x^2 + 11x - 10$

Looking for two numbers whose product is –10 × 6 i.e. –60 and sum is 11, we zero down on 15 and –4. Thus,

$6x^2 + 11x - 10 = 6 \times \left(x+\frac{15}{6}\right)\times\left(x-\frac{4}{6}\right)$ and the roots are $-\frac{15}{6}$ and $\frac{4}{6}$ i.e. $-\frac{5}{2}$ and $\frac{2}{3}$

Practice

Factorise the following (Answer at end of this topic)

1. $5x^2 + 27x + 10 =$ ___ $\times (x \qquad) \times (x \qquad)$
2. $2x^2 + 19x + 30 =$ ___ $\times (x \qquad) \times (x \qquad)$
3. $3x^2 - 11x + 6 =$ ___ $\times (x \qquad) \times (x \qquad)$
4. $3x^2 - 10x + 8 =$ ___ $\times (x \qquad) \times (x \qquad)$
5. $3x^2 + 34x + 11 =$ ___ $\times (x \qquad) \times (x \qquad)$

Finding roots by Formula

Please note that not all equations can be solved by the process of factorization and if the equation cannot be factorised, one would have to use the formula. The formula is:

Roots of the quadratic equation $ax^2 + bx + c = 0$ is $\dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

E.g. 12: The sum of squares of three consecutive natural numbers is 302. What is the middle number?

Let the three numbers be x, $(x + 1)$ and $(x + 2)$.

Thus, $x^2 + (x + 1)^2 + (x + 2)^2 = 302$

$x^2 + x^2 + 2x + 1 + x^2 + 4x + 4 = 302$

$3x^2 + 6x - 297 = 0$

$x^2 + 2x - 99 = 0$

$(x + 11)(x - 9) = 0$

$x = -11$ or 9.

Since x is a natural number, $x = 9$.

Thus the middle number is 10.

Determinant and Nature of the Roots

The expression under the radical (root) sign i.e. '$b^2 - 4ac$' is called the Determinant of the equation and is denoted as D. Thus $D = b^2 - 4ac$. The determinant of the equation is used to identify the Nature of the roots, as follows:

Determinant	Nature of Roots
$D < 0$ i.e. D is negative	Roots are imaginary
$D = 0$	Roots are real and equal
$D > 0$	Roots are real and unequal

E.g. 13: For what value of b will the equation $x^2 + bx - 9 = 0$ have equal roots?

Roots of a quadratic equation are equal when $b^2 - 4ac = 0$

Thus $b^2 - 4 \times 1 \times (-9) = 0$ i.e. $b^2 = 36 \Rightarrow b = \pm 6$

Thus when b = 6 or –6, the equation will have equal roots.

Sum and Product of the roots

To find the sum and product of roots, one need not find the roots and then add or multiply them. One can directly find the sum and product of the roots by just knowing the coefficients of the x^2, x, and the constant term using the following formulae:

For the quadratic equation $ax^2 + bx + c = 0$,

Sum of the roots = $-\dfrac{b}{a}$

Product of the roots = $\dfrac{c}{a}$

Most of the questions on quadratic equation in entrance exams are based on the above relations. So learn them thoroughly and also understand the following examples thoroughly.

E.g. Find the sum of the roots of the equation $10 + 6x - 2x^2 = 0$

Using the formula, the sum of the roots is $-\frac{6}{-2} = 3$

E.g. 14: If one root of the equation $x^2 - 6x + c = 0$ is double the other root, find the value of c.

Since one of the root is double of the other, we can assume the roots as p and $2p$.

Also from the given equation, we can know that the sum of the roots is 6.

Thus $p + 2p = 6$ i.e. $3p = 6$ i.e. $p = 2$.

Thus the two roots are 2 and 4

Now the product of the roots is c because $a = 1$. Thus $c = 2 \times 4 = 8$.

E.g. 15: If the roots of the equation $x^2 - (k + 3)x + k = 0$ are reciprocals of each other, find the sum of the roots.

Since the roots of the equation are reciprocals of each other, the product of the roots is 1.

Thus $\frac{c}{a} = \frac{k}{1} = 1 \Rightarrow k = 1$

Sum of the roots = $-\frac{b}{a} = -\frac{-(k+3)}{1} = (k+3) = 4$

Forming the equation when the roots are given

If the roots of a quadratic equation are p and q, then the equation can be constructed as follows:
$x^2 - (p + q)x + pq = 0$

In other words a quadratic equation can be constructed as follows:

x^2 – (sum of the roots)x + product of the roots = 0

E.g. 16: Find the equation whose roots are 3 and –7.

Sum of roots = 3 + (–7) = –4

Product of the roots = –21

Thus the equation is $x^2 - (-4)x + (-21) = 0$

$x^2 + 4x - 21 = 0$

E.g. 17: If p and q are the roots of the equation $x^2 - 2x + 3 = 0$, find the equation whose roots are $(p + 3)$ and $(q + 3)$.

In such questions it would not be the right strategy to find the root of the given equation because most probably the equation would not be factorisable and the roots would be imaginary or irrational.

The way out is as follows:

We need to find the equation: $x^2 - (p + 3 + q + 3)x + (p + 3)(q + 3) = 0$

i.e. $x^2 - (p + q + 6)x + (pq + 3p + 3q + 9) = 0$

Since p and q are the roots of $x^2 - 2x + 3 = 0$, we already know that $p + q = 2$ and $pq = 3$.

Thus, the required equation is $x^2 - (2 + 6)x + (3 + 3\times2 + 9) = 0$ i.e. $x^2 - 8x + 18 = 0$

Exercise

17. Find the solution to $(x - 5)^2 = 4(x - 5)$

    a. 5 b. 9 c. 5 or 9 d. None of these

18. A packet of goods costs Rs. 360. If the quantity in the packet was 6 kgs more and the rate per kg had been Rs. 3 less, the total cost would still have been 360. How many kgs did the original packet contain?

    a. 30 b. 24 c. 15 d. 18

19. 23 is divided into two parts such that the difference between the squares of each part is 23. Find the two parts.

    a. 8, 15 b. 10, 13 c. 12, 11 d. 9, 14

20. For what value of c would the roots of the equation $x^2 - 6x + c = 0$ be real?

    a. $c \le 6$ b. $3 \ge c$ c. $c \le 9$ d. $c \ge 9$

21. For what value of k would the roots of the equation $(k - 1)x^2 + (2k + 2)x + k = 0$ be equal?

    a. 1 b. 1/3 c. – 1/3 d. None of these

22. Find the value of $1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\ldots\ldots}}}$.

    a. $\frac{1\pm\sqrt{5}}{2}$ b. $\frac{1+\sqrt{5}}{2}$ c. $\frac{1-\sqrt{5}}{2}$ d. $\frac{1+\sqrt{3}}{2}$

23. If one root of the equation $x^2 - bx - 8 = 0$ is square of the other root, find the value of b?

    a. 2 b. 6 c. 8 d. 10

24. If the roots of the equation $x^2 - 3x + 4 = 0$ is p and q, find the equation whose roots are $(p + 2)$ and $(q + 2)$.

    a. $x^2 - 7x + 14$ b. $x^2 - 7x + 12$ c. $x^2 + 7x - 14$ d. $x^2 + 7x - 12$

Answers to Practise Questions:

1. $(x + 1) \times (x + 9)$ 2. $(x + 4) \times (x + 3)$ 3. $(x - 4) \times (x + 2)$ 4. $(x - 7) \times (x + 1)$ 5. $(x + 4) \times (x - 1)$
6. $(x + 6) \times (x - 2)$ 7. $(x - 5) \times (x - 5)$ 8. $(x + 3) \times (x + 6)$ 9. $(x - 12) \times (x + 2)$ 10. $(x + 11) \times (x + 12)$

1. $5\times\left(x+\frac{25}{5}\right)\times\left(x+\frac{2}{5}\right)$ i.e. $(x + 5) \times (5x + 2)$
2. $2\times\left(x+\frac{15}{2}\right)\times\left(x+\frac{4}{2}\right)$ i.e. $(2x + 15) \times (x + 2)$
3. $3\times\left(x-\frac{9}{3}\right)\times\left(x-\frac{2}{3}\right)$ i.e. $(x - 3) \times (3x - 2)$
4. $3\times\left(x-\frac{6}{3}\right)\times\left(x-\frac{4}{3}\right)$ i.e. $(x - 2) \times (3x - 4)$
5. $3\times\left(x+\frac{33}{3}\right)\times\left(x+\frac{1}{3}\right)$ i.e. $(x + 11) \times (3x + 1)$

Progressions

Progressions are series of numbers, where each successive number in the series is derived from the previous number according to a rule. Two most popular progressions are Arithmetic Progression (*A.P.*) and Geometric Progression (*G.P.*)

Arithmetic Progression

In an *A.P.*, the successive term is got by adding a constant value to the last term. Each *A.P.* is characterized by its first term and by the constant that is added e.g. if the first term is 15 and the constant to be added is 4, we get the following series:

15, 19, 23, 27, 31, 35, 39, ……

Thus we see that the difference between the successive terms of an *A.P.* is a constant (the constant that is added)

The first term of an *A.P.* is denoted by a and the constant difference between the successive terms is called 'common difference' and is denoted by d.

The value of any n^{th} term is denoted by T_n and one can find the value of any n^{th} term using the formula:

$T_n = a + (n - 1)d$

The above formula can be re-arranged to find the number of terms as follows:

$$n = \frac{\text{Last Term} - \text{First Term}}{d} + 1$$

The only other formula used in *A.P.* is the formula to find the sum of first n terms of an *A.P.*

$$S_n = \frac{n}{2}\{2a + (n-1)d\} \text{ or } S_n = \frac{n}{2}(\text{First Term} + \text{Last Term})$$

If the first and the last term is known you can use the second formula, but you will yet need to know the number of terms.

E.g. 18: What is the 25th term of the series –23, –16, –9, –2, 5, 12…

Here $a = -23$ and $d = 7$ and we need to find the 25^{th} term i.e. T_{25}. Thus substituting $n = 25$ in the formula for T_n, we get

$T_{25} = -23 + 24 \times 7 = -23 + 168 = 145.$

E.g. 19: Find the number of terms in the series: 32, 29, 26, 23, ….., –19.

This is an *A.P.* with the first term being 32, last term being –19 and d = –3.

Thus the number of terms = $\frac{-19 - 32}{-3} + 1 = \frac{-51}{-3} + 1 = 18$

E.g. 20: Find the sum of first 10 terms of the series 11, 21, 31, ….

This is an *A.P.* with $a = 11$ and $d = 10$ and we need to find S_{10}. Thus putting the value of $n = 10$ in the formula for the sum of first n terms of *A.P.*, we get,

$$S_{10} = \frac{10}{2}(2 \times 11 + 9 \times 10)$$

$$S_{10} = 5(22 + 90) = 5 \times 112 = 560$$

E.g. 21: Find the sum of all terms of the series $\frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, ..., \frac{31}{2}$

If we find the difference between any two terms we see that it is always $\frac{1}{2}$. Thus the series given is an *A.P.* Since we know the first term and the last term, lets find the number of terms so that we can use the formula $S_n = \frac{n}{2}(\text{First Term} + \text{Last Term})$ to find the sum of the terms.

The number of terms = $\frac{\text{Last Term} - \text{First Term}}{d} + 1 = \frac{\frac{31}{2} - \frac{1}{2}}{\frac{1}{2}} + 1 = \frac{\frac{30}{2}}{\frac{1}{2}} + 1 = 31$

The sum of the terms = $\frac{31}{2}\left(\frac{1}{2} + \frac{31}{2}\right) = \frac{31}{2} \times \frac{32}{2} = 248$

Geometric Progression

An *A.P.* was got by adding a constant to the last term to get the successive terms. A *G.P.* is got by multiplying the last term by a constant to get the next term. The *G.P.* is also characterized by the first term, denoted by a and the constant with which each term is multiplied to get the next term. This constant is called common ratio and is denoted by r.

If $a = 2$ and $r = 3$, we get the *G.P.*: 2, 6, 18, 54, 162, ...

If $a = 32$ and $r = \frac{1}{2}$, we get the *G.P.*: 32, 16, 8, 4, 2, 1, $\frac{1}{2}$, $\frac{1}{4}$, ...

Just as in an *A.P.*, we have formulae for finding the value of any n^{th} term and for finding the sum of first n terms of a *G.P.*

$T_n = ar^{(n-1)}$

$S_n = a\frac{r^n - 1}{r - 1}$

Infinite GP with r < 1

In the case of a *G.P.* there is a specific *G.P.* where r is less than 1 (to be accurate r has to be between –1 and 1) and the *G.P.* has infinite terms, e.g.

6, 2, $\frac{2}{3}$, $\frac{2}{9}$, $\frac{2}{27}$, ...

Since we are multiplying by r, which is less than 1, the terms of this *G.P.* become smaller and smaller as the number of terms keeps increasing. Quite a few of the *G.P.* questions are based on such a *G.P.*. For such a decreasing *G.P.* (r being less than 1), the sum of infinite terms is given by $S_\infty = \frac{a}{1 - r}$

E.g. 22: If the first term of a *G.P.* is $\frac{1}{256}$ and the common ratio is –4, find the 10^{th} term of the *G.P.*

$T_{10} = \frac{1}{256} \times (-4)^{(10-1)} = \frac{1}{4^4} \times (-4)^9 = -4^5 = -1024$

E.g. 23: What is the sum of the series 16, 8, 4, 2, 1, $\frac{1}{2}$, ...

This is an infinite *G.P.* with r being less than 1 and hence we use the formula $S_\infty = \frac{a}{1-r}$ and get the sum of infinite terms = $\frac{16}{1-\frac{1}{2}} = 32$

E.g. 24: What is the sum of the series 16, –8, 4, –2, 1, $-\frac{1}{2}$, ...

This is an infinite *G.P.* and since with r is between –1 and 1, we can use the formula $S_\infty = \frac{a}{1-r}$ and get the sum of infinite terms = $\frac{16}{1-\left(-\frac{1}{2}\right)} = \frac{16}{\frac{3}{2}} = \frac{32}{3}$

E.g. 25: A rubber ball is thrown up and it reaches a height of 64 feets and then falls back. On each bounce it bounces back to $\frac{3}{4}^{th}$ of the earlier distance. Find the distance traveled by the ball till it comes to rest.

The distances that the ball travels the first time and then on each bounce is 64, 64 × $\frac{3}{4}$ = 48, 48 × $\frac{3}{4}$ = 36, 36 × $\frac{3}{4}$ = 27 and so on.

Thus the distances traveled are in a decreasing *G.P.* with first term being 64 and common ration being $\frac{3}{4}$.

The sum of the *G.P.* 64, 48, 36, 27, ... is $\frac{64}{1-\frac{3}{4}} = 256$.

Now the ball travels each of the distances 64, 48, 36, 27, ... twice, once while going up and once while falling down. Thus the total distance traveled by the ball till it comes to rest is 2 × 256 = 512.

Exercise

25. Find the sum of first 35 terms of the series: 51 + 48 + 45 +……….

    a. 102 b. 0 c. 84 d. 92

26. Find the number of terms in the following series: 7 + 14 + 21 + 28 + ………… + 147

    a. 20 b. 22 c. 21 d. 23

27. Find the 100^{th} term of the sequence 14, 18, 22, 26, ……..

    a. 400 b. 410 c. 200 d. 205

28. If the 5^{th} term of an *A.P.* is 8 and 7^{th} term is 16, find the 13^{th} term of the same series?

    a. 24 b. 26 c. 40 d. None of these

29. Find the sum of first 15 terms of an *A.P.* whose 6^{th} and 10^{th} terms are 10 and 6 respectively.

    a. 120 b. 125 c. 140 d. 150

30. What is the sum of first 50 odd numbers?

    a. 1200 b. 2500 c. 2400 d. 1000

31. If in a *G.P.*, the sixth term is $-\frac{1}{9}$ and the ninth term is $\frac{1}{243}$, find the first term?

    a. 27 b. 9 c. 18 d. 24

32. Find the sum of the infinite *G.P.* 90, 60, 40, ………

    a. 180 b. 270 c. 900 d. 360

33. If the first term of a *G.P.* is 120 and the common ratio is $\frac{3}{4}$, then find the 4^{th} term of this series.

    a. 205/8 b. 20 c. 405/8 d. 50

34. If the 8^{th} and 10^{th} terms of the geometric series are $\frac{1}{32}$ and $\frac{1}{128}$ respectively, then find the first term of the series.

    a. 2 b. 4 c. 8 d. 16

35. The 11^{th} and 17^{th} terms of an *A.P.* are 11 and 8 respectively. Starting with the first term, how many terms of the series need to be added such that the sum is 231.

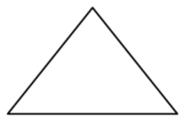
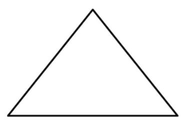
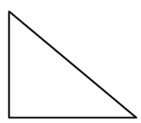
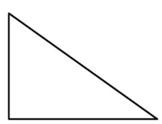
    a. 12 b. 35 c. 77 d. 70

36. E_1 is an equilateral triangle whose area is 16 sq. cm. The mid-points of the sides of the triangle are joined to form another triangle E_2. The mid-points of the sides of E_2 are joined to form another triangle E_3 and the process is repeated infinite times. Find the sum of the areas of all triangles E_1, E_2, E_3, …

    a. 32 b. 64/3 c. 64 d. None of these

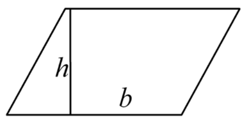
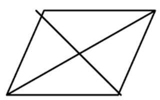
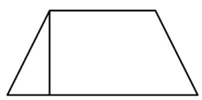
Mensuration

Triangles

No.	Name	Figure	Perimeter	Area
		TRIANGLES		
1.	Any Triangle		$a+b+c$	1. $\frac{1}{2}\times b\times h$ where b is the base and h the height. 2. $\sqrt{s(s-a)(s-b)(s-c)}$ s is semi-perimeter and is equal to $\frac{a+b+c}{2}$
2.	Equilateral Triangle		$3a$ (each side is a) Altitude = $\frac{\sqrt{3}}{2}a$	$\frac{\sqrt{3}}{4}a^2$
3.	Isosceles Right Triangle		$2a+d$ a – base & height d – hypotenuse $d=\sqrt{2}a$	$\frac{a^2}{2}$ base & height are equal to a
4.	Right Triangle		$b+h+d$ b – base h – height d – hypotenuse $b^2+h^2=d^2$	$\frac{1}{2}\times b\times h$

Further, for any triangle, the sum of all three angles is always 180 degrees.

Quadrilaterals

No.	Name	Figure	Perimeter	Area
		QUADRILATERALS		
1.	Rectangle		$2(l+b)$ l – length b – breadth	$l \times b$
2.	Square		4a a – side	a^2
3.	Parallel-ogram		$2(a+b)$ a and b are sides.	$b \times h$ h is the distance between the parallel sides.
4.	Rhombus		$4a$ All the sides are equal to a	$\frac{1}{2} \times d_1 \times d_2$ d_1 and d_2 are diagonals
5.	Trapezium		Sum of its four sides is its perimeter.	$\frac{1}{2} \times (a+b) \times h$ h – perpendicular distance between the parallel sides a and b are parallel sides

Hardly any questions are asked on the above plane figures. Usually the questions are on solids. But nevertheless it is a good idea to know the above formulas for perimeter and area.

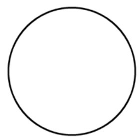
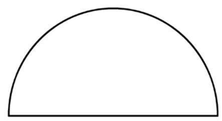
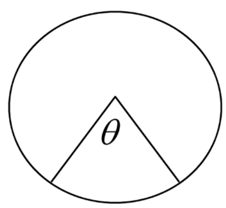
E.g. 1: The area of a rhombus is 240 cm^2 and one of the diagonals is 16 cm. Find the other diagonal.

Using the formula for area of a rhombus, we have $\frac{1}{2} \times 16 \times d_2 = 240 \Rightarrow d_2 = 30$

E.g. 2: The area of a trapezium is 34 cm^2 and the length of one of the parallel sides is 10 cm and its height is 4 cm. Find the length of the other parallel side.

Using the formula for area of a trapezium, we have $\frac{1}{2} \times (10 + l_2) \times 4 = 34 \Rightarrow l_2 = 7$

Circles

No.	Name	Figure	Perimeter	Area
		CIRCLES		
1.	Circle		$2\pi r$ r is the radius.	πr^2
2.	Semicircle		$\pi r + 2r$	$\frac{\pi r^2}{2}$
3.	Ring (shaded region)			$\pi\left(R^2 - r^2\right)$ R – outer radius r – inner radius
4.	Sector of a circle		$\frac{\theta}{360} \times 2\pi r + 2r$	$\frac{\theta}{360} \times \pi r^2$

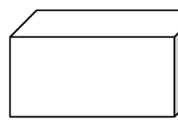
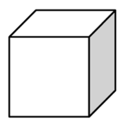
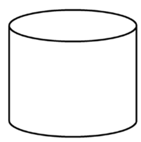
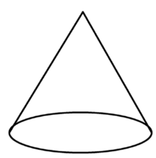
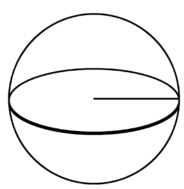
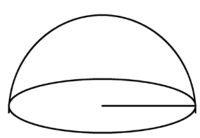
E.g. 3: A road roller takes 750 complete revolutions to move once over to level a road. Find the area of the road, in m^2, if the diameter of a road roller is 84 cm and length is 1 m.

In one revolution the road roller will cover a distance equal to its circumference i.e. $2 \times \frac{22}{7} \times 42$ = 12 × 22 cm.

In one revolution, the area that is levelled will be this distance multiplied with the length of the roller i.e. in 1 revolution the area levelled will be 12 × 22 × 100 cm^2.

Hence in 750 revolutions the area levelled will be 12 × 22 × 100 × 750 cm^2. Changing it to m^2, we get the answer as $\frac{12 \times 22 \times 100 \times 750}{100 \times 100}$ = 12 × 11 × 15 = 1980 m^2

Solids

S. No	Name	Figure	Lateral/Curved Surface Area	Total Surface Area	Volume
1.	Cuboid		$2 \times h \times (l + b)$ h – height l – length b – breadth	$2(lb + bh + lh)$	$l \times b \times h$
2.	Cube		$4a^2$ a – edge	$6a^2$	a^3
3.	Right Circular Cylinder		$2\pi rh$ r – radius h – height	$2\pi(r + h)$	$\pi r^2 h$
4.	Right Circular Cone		πrl h – height r – radius l – slant height $l^2 = r^2 + h^2$	$\pi r(r + l)$	$\frac{1}{3} \times \pi r^2 h$
5.	Sphere		$4\pi r^2$ r – radius	$4\pi r^2$ r – radius	$\frac{4}{3} \times \pi r^3$
6.	Hemi-sphere		$2\pi r^2$	$3\pi r^2$	$\frac{2}{3} \times \pi r^3$

E.g. 4: The internal measures of a cuboidal room are 12 m × 8 m × 4 m, where 4 m is the height. Find the total cost of whitewashing all four walls of a room and the ceiling, if the cost of white washing is Rs 5 per m^2.

The area of the two pair of opposite walls will be 12 m × 4 m and 8 m × 4 m.

Thus, the area of the four walls will be 2 × (48 + 32) = 160 m^2

The area of the ceiling will be 12 m × 8 m = 96 m^2

Thus, total area to be whitewashed = 160 + 96 = 256 m^2 and the cost will be 256 × 5 = Rs. 1280.

E.g. 5: In a building there are 25 cylindrical pillars. The radius of each pillar is 28 cm and height is 4 m. Find the total cost of painting the curved surface area of all pillars at the rate of Rs 8 per m^2.

The curved surface area of each pillar = $2\times\frac{22}{7}\times\frac{28}{100}\times 4=\frac{176}{25}$ m^2.

Thus, total cost for painting 25 pillars will be $\frac{176}{25}\times 25\times 8$ = Rs. 1408.

E.g. 6: A godown is in the form of a cuboid of measures 60 m × 40 m × 30 m. How many cuboidal boxes can be stored in it if the volume of one box is 0.8 m^3?

The number of boxes that can be fitted in = $\frac{60\times 40\times 30}{0.8}$ = 90,000

E.g. 7: A joker's cap is in the form of a right circular cone of base radius 7 cm and height 24 cm. Find the area of the sheet required to make 10 such caps.

To find the curved surface area of a cone, we need the slant height.

If l denotes the slant height of the cone, we have $l^2 = 7^2 + 24^2 = 49 + 576 = 625$ i.e. $l = 25$.

Curved surface area of the cap = $\frac{22}{7}\times 7\times 25 = 550$ cm^2

E.g. 8: A hemispherical bowl has a radius of 3.5 cm. What would be the volume of water it would contain?

The volume of the hemi-sphere will be half that of the sphere and hence will be $\frac{1}{2}\times\frac{4}{3}\times\frac{22}{7}\times\left(\frac{7}{2}\right)^3=\frac{539}{6}$ cm^3.

E.g. 9: The diameter of a sphere is decreased by 25%. By what per cent does its curved surface area decrease?

Consider the spheres having radius 4. After the reduction, its radius will be 3 (if diameter is 25% less, then radius will also be 25% less). The original surface area = $4\pi\times 16$ and the reduced surface area will be $4\pi\times 9$. Thus the percentage reduction will be $\frac{16-9}{16}\times 100=\frac{7}{16}\times 100=\frac{175}{4}$ = 43.75%

Exercise

1. The ice compartment of a refrigerator is 8 inches long, 4 inches wide and 5 inches high. How many ice cubes will it hold if it each cube's edge is 2 inch?

   a. 16 b. 20 c. 12 d. 15

2. The volume of a rectangular box with a square base is 6 cubic feet. If the height of the box is 18 inches find the side of the square base?

   a. 1 feet b. 2 feet c. 20 inches d. 12 inches

3. If cylinder A has 3 times the height and one-third the diameter of cylinder B, what is the ratio of the volume of A to volume of B?

   a. 3 : 1 b. 1 : 3 c. 1 : 9 d. 1 : 1

4. There are two spherical balls each having a radius of 4 metre. Both of them are melted and a bigger ball is made, find the radius of this new ball.

a. $4\sqrt{2}$ m b. $4\times\sqrt[3]{2}$ m c. 8 m d. $\sqrt{8}$ m

5. There is a cylindrical pipe, whose height is 20 metre, and outer radius is 2 m and inner radius is 1.5 m. Find out the volume of the metal used in the pipe.

a. 150 cubic m b. 120 cubic m c. 110 cubic m d. 125 cubic m

6. Three cubes of sides 4 cm, 5 cm and 6 cm are melted together to form a bigger cube. Find the side of the new cube?

a. $3\times\sqrt[3]{15}$ b. $2\times\sqrt[3]{15}$ c. $3\sqrt{2}$ d. $2\sqrt{3}$

7. A spherical copper ball whose diameter is 14 cm, is melted and converted into a wire having diameter 14 cm. Find the length of the wire.

a. 28 cm b. 14/3 cm c. 28/3 cm d. 14 cm

8. If the height and radius of the base of a cylinder are both increased by 200%, find the increase in volume of the cylinder.

a. 25 times b. 8 times c. 27 times d. 26 times

9. What is the volume of a cube whose largest diagonal is $8\sqrt{3}$ cm.

a. 500 cubic cm b. 508 cubic cm c. 600 cubic cm d. 512 cubic cm

10. Two cones of equal volumes have their radii in the ratio 2:3. Find the ratio of their heights.

a. 3:2 b. 2:3 c. 9:4 d. 4:9

11. Find the volume of a cube whose surface area is 150 sq. cm.

a. 125 cubic cm b. 64 cubic cm c. 216 cubic cm d. 27 cubic cm

12. There is a conical tomb whose slant height is 13 m and diameter is 10 m. find the cost of construction of the tomb at the rate of Rs. 2 per cubic metre.

a. Rs. 600 b. Rs 528 c. Rs. 628.57 d. Rs. 514.50

13. If a cone, a hemisphere and a cylinder stand on the same base and have the same height, find the ratio of their volumes?

a. 1:3:2 b. 2:3:1 c. 1:2:3 d. 3:2:1

14. A cylindrical vessel with radius of base 14 cm is half filled with water. Three solid metallic spheres of radius 7 cm are put in the cylindrical vessel and all the three spheres sink in the water. By what height would the level of water in the cylindrical vessel increase, if no water spills out?

a. 7 mm b. 10 cm c. 7 cm d. Cannot be determined

15. A field is in the form of a rectangle having length 20 m and width of 15 m. In the middle of the field, a square well of side 10 m is dug and the soil taken out by digging is uniformly spread over the remaining field. If the well is dug to a depth of 10 mts, by what amount does the level of the field increase because of spreading the dug-out soil over the field?

a. 4 m b. 3 m c. 6 m d. 5 m

Assignment: Algebra & Mensuration

1. On being asked her age, a girl says "my age five years from now would be 1.5 times my age five years back". What will be her age ten years from now?

   (a) 20 years (b) 35 years (c) 25 years (d) 30 years

2. A third of Raju's marks in Math exceeds a half of his marks in English by 30. If his total score in both the subjects is 240, his marks in Math and English are:

   (a) 60, 180 (b) 180, 60 (c) 200, 160 (d) 160, 200

3. A shopkeeper sell half of the rice with him and donates half a kilo. He again sells half the remaining rice and donates half a kilo. Now he has no rice left in his store. How much rice did he have in his store initially?

   (a) 4 kg (b) 3 kg (c) 5 kg (d) 2 kg

4. In a parking lot of a park, there are either 3 wheelers or 8 wheelers. If there are 100 vehicles in all and 500 wheels, then how many three wheelers are there?

   (a) 60 (b) 40 (c) 80 (d) 70

5. If two pencils and 3 pens cost Rs. 19 and 5 pens and 3 pencils cost Rs. 21, what is the price of 5 pencils and 8 pens?

   (a) Rs. 25 (b) Rs. 30 (c) Rs. 35 (d) Rs. 40

6. An employer pays Rs. 20 for each day of work and forfeits Rs. 3 for each idle day spent. At the end of 60 days, a worker gets Rs. 280. For how many days was the worker?

   (a) 30 (b) 20 (c) 40 (d) 15

7. A total sum of Rs. T was to be given to 18 workers but 4 of which did not turn up for collecting their share of salary. Hence the extra amount left over was distributed equally among the others, thereby increasing their salaries by Rs. 80. What is the value of T?

   (a) 5000 (b) 5040 (c) 2060 (d) 3080

8. The difference between a two digit number and the number formed by reversing the two digits is 27. Which of the following can be the product of the two digits?

   (a) 8 (b) 10 (c) 7 (d) none of these

9. If from a positive natural number, we subtract 4 times its reciprocal, the result is –3. What is the number?

   (a) –4 (b) 1 (c) 2 (d) 4

10. If 4 times a positive number is subtracted from square of the natural number, the result is 21. What will be the result if a similar operation is done on a number two more than the earlier number?

    (a) 18 (b) 16 (c) 5 (d) 45

11. A wheel of a car, having radius equal to 21 cms, is rotating at 600 RPM (revolutions per minute). What is the speed of the car in km/hr?

    (a) 23.76 kmph (b) 47.52 kmph (c) 37.25 kmph (d) 28.95 kmph

12. What can be the maximum area of circle that can be placed in a rectangle with 20 × 10 dimensions (in cm^2) ?

    (a) 25π (b) 50π (c) 400π (d) 100π

13. A cylindrical jar with base radius 21 cm has some water in it. How many marbles of diameter 14 cm are required to be put in to the jar to raise the water level by 56 cm?

    (a) 378 (b) 126 (c) 210 (d) 248

14. A horse was tied to a pole by a rope of length 20 mtrs. If the length of the rope is increased to 30 mtrs then what is the ratio of increase in the area that the horse can graze to the previous area available for grazing?

    (a) 1.5 (b) 0.75 (c) 1.25 (d) 0.66

15. A cube with side 6 cm was cut in to smaller cubes with side 3 cm each. What is the percentage increase in the surface area?

    (a) 100% (b) 200% (c) 300% (d) 400%

16. In a cylinder, if radius is halved and height is doubled, the volume will be

    (a) same (b) doubled (c) halved (d) four times

17. A cone is 8.4 cm high and the radius of its base is 2.1 cm. It is melted and recast into a sphere. The radius of the sphere is:

    (a) 4.2 cm (b) 2.1 cm (c) 2.4 cm (d) 1.6 cm

18. A cone like tent is to be formed by using cloth. How many meters are required if height of the tent is 4 mtr and base radius is 3 mtrs?

    (a) 10π (b) 12π (c) 15π (d) 20π

19. A sphere of radius 5 cm is placed inside a cube of side 10 cm. Find the ratio of volume left over inside the cube to the volume of sphere.

    (a) 9.09 (b) .909 (c) 1.909 (d) 1

20. A rocket is formed by placing a cone with radius 2 cm and height 3 cm as the tip on a cylinder with height 15 cm and radius 1 cm. Find the ratio of volume of tip of the rocket to volume of entire rocket.

    (a) 12/27 (b) 20/27 (c) 8/21 (d) 12/15

Permutation and Combination

Permutation & Combination – it invariably evokes groans from a lot of students. And ironically the topic is just about counting, counting the number of ways in which certain event can happen. Since the counting can extend to a large number, we would fall short of fingers to count with and hence certain rules are framed. So P & C is essentially about rules to help us count.

Fundamental Rules of Counting

Rule of *AND*

If event A can happen in m different ways and event B can happen in n different ways, event *A and B* can happen in $m \times n$ ways.

If I have 3 different shirts and 2 different trousers, the total number of different ways in which I can wear a shirt and a trouser is 3 × 2 = 6 ways.

The rule is not limited to just two events. In the above case if I also had 2 different ties and I had to also wear a tie, then just with shirts and trousers, we already had 6 different possibilities. Now *with each of above* 6 possibilities, *there are further* 2 ways to select a tie and thus in all the possibilities would be 6 × 2 = 12.

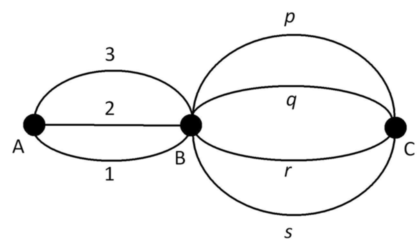
The above would look obvious, because it is shirts, trousers, ties and we regularly mix and match. What about questions like “How many 3 digit numbers can one form using 1, 2, 3, 4 and 5, each digit being used only once?”

Well this question is EXACTLY same as the above. Just as we have a fixed place for wearing shirts, trousers and ties, in a three digit number also we have three places – units, tens and hundreds. And instead of 2 trousers, 3 shirts and 2 ties, here we have to use the digits 1, 2, 3, 4 and 5 (the only difference being that any of the digits can go in any place unlike the fact that you cannot wear a shirt in place of trouser).

Starting with the hundreds position, it can be filled in 5 ways, any of 1, 2, 3, 4, or 5. In each of these 5 ways, one of the digits out of the 5 given digits has been used and cannot be used again. Thus, f*or each of these above* 5 ways, *there are further* 4 ways in which the ten’s digit can be filled – any of the remaining 4 digits could be filled in the ten’s position. Thus we get 5 × 4 = 20 different ways in which the hundred’s and ten’s digit can be filled. Further in each of these 20 ways, two digits out of the available five digits would have been used and cannot be used again in filling the units place. Thus *for each of the above* 20 ways, *there are further* 3 ways to fill the units place. Thus, the total possible numbers formed will be 20 × 3 = 60.

Using the rule of *AND*: the hundreds position can be filled in 5 ways; having filled the hundreds position, the tens position can be filled in 4 ways; and having done this, the units position can be filled in 3 ways. Since we have to fill the hundreds position *AND* the tens position *AND* the units position, the total number of ways is 5 × 4 × 3 = 20.

E.g. 1: There are 3 ways to travel from A to B and 4 ways to travel from B to C.



1. In how many different ways can one travel from A to C?

2. In how many different ways can one travel from A to C and come back to A?

3. In how many different ways can one travel from A to C and come back to A travelling on a different road between each of A to B and B to C while going and while coming back?

1. From A to B one can choose a way in 3 different ways and from B to C one can choose a way in 4 different ways. Since one has to choose a way from A to B *AND* B to C, the total number of different ways is 3 × 4 = 12.

Using simple logic, one would be able to identify the 12 ways as:

1-*p*   1-*q*   1-*r*1-*s*2-*p*   2-*q*   2-*r*2-*s*3-*p*   3-*q*   3-*r*3-*s*

2. A to B can be travelled in 3 ways; B to C in 4 ways; back C to B in 4 ways; and back B to A in 3 ways. Thus total number of ways is 3 × 4 × 4 × 3 = 144.

Logically, f*or each of the above* 12 ways of going from A to C, *there are further* 12 ways of returning from C to A.

Thus the total number of ways of the round trip will be 12 × 12 = 144.

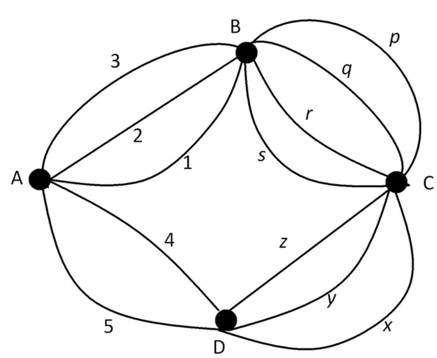
3. From A to B we can travel by any of the 3 ways; from B to C by any of the 4 ways; on way back from C to B by any of the 3 ways (the way taken from B to C cannot be taken again); from B to A by any 2 way (the way taken from A to B cannot be taken again).

Thus, total number of ways is 3 × 4 × 3 × 2 = 72 ways.

Rule of *OR*

If event *A* can happen in *m* different ways and event *B* can happen in *n* different ways, event *A or B* can happen in *m* + *n* ways.

E.g. In the example of travelling from A to C, let's say the road network was as shown below:



And we are supposed to find the number of ways in which we can travel from A to C.

If we start with – there are 5 ways to proceed from A (viz. 1, 2, 3, 4, 5), then we would not be able to process further. It would be wrong to say that the total number of ways would be 5 × 4 or 5 × 3 or 5 × 7. After the first leg of journey, which can be done in 5 ways, there are not 4 ways (or 3 ways or 7 ways) for *each of the earlier* 5 ways.

In this case we would need to break the event of travelling from A to C in two distinct cases one via B and one via D. Thus I could travel A-B-C *or* A-D-C.

A-B-C can be travelled in 3 × 4 = 12 ways and A-D-C can be travelled in 2 × 3 = 6 ways.

Since I can travel on only one of these two paths (A-B-C *or* A-D-C), the total number of ways is 12 + 6 = 18. (Remember in this case *for each of the* 12 ways from A-B-C, there ARE NOT *further* 6 ways. Travelling along these two routes is not sequential, one after the other, it is a case of *or*)

E.g. 2: How many natural numbers can be formed using any or all of the digits 1, 2, 3, 4, each digit being used maximum once.

The only difference between this question and the example explained earlier is that in this question the number of digits in the number to be formed is not specified. Thus we can form a single digit number *or* a two-digit number *or* a three-digit number *or* a four digit number. We cannot form a number with digits being more than 4 since each of the given number can be used just once.

Number of single digit numbers that can be formed: 4

Number of two-digit numbers that can be formed: 4 × 3 = 12 (the ten's position can be filled with any of the four digits and *for each of these* 4 ways, the unit's digit can be filled *further in* 3 ways).

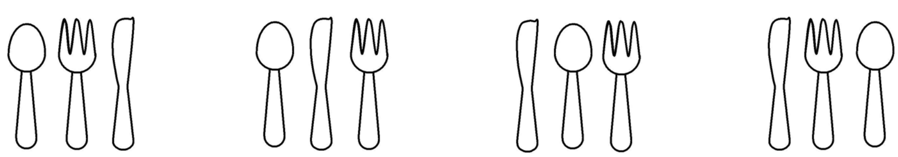
Number of three-digit numbers that can be formed: 4 × 3 × 2 = 24

Number of four-digit numbers that can be formed: 4 × 3 × 2 × 1 = 24

Thus, the total possible natural numbers that can be formed = 4 + 12 + 24 + 24 = 64. (Now we are adding and not multiplying because we can form a single digit *or* two-digit *or* three-digit *or* four-digit)

Arrangements (Permutation)

Arrangements are characterised by 'specific way in which objects are arranged'. The word 'arranged' suggests a specific 'order' among the objects being arranged. E.g. if I have to arrange a fork, a knife and a spoon on the table, each of the following is a different arrangement:



Way 1 Way 2 Way 3 Way 4

Each of the above depicted way is different is because of 'order' of placing the three objects is different. When we say 'order' we refer to the relative placement of the objects among themselves.

The above example is just to make you understand what we mean by 'arrangements' or the importance of 'order'. Arranging objects would thus mean placing objects in specific 'position'.

Are these ALL the possible ways of arranging the three objects? It should easily be possible for you to come up with more ways and thus these are not the only way. So what is the total number of possible ways of arranging the three objects among themselves?

We have three positions and three objects. The first position can be filled in three ways with any of the fork, knife or spoon occupying it. Having filled this, *for each of the 3 ways*, the second position can be *further filled in 2 ways*, with either of the two remaining objects. Having filled this, the third position can be filled in only one way with the object remaining. Since we have to place object in first position *and* second position *and* the third position, total number of ways is 3 × 2 × 1 = 6 ways.

Thus, arrangement of objects in positions primarily uses the same funda as Rule of *AND*.

It does not matter what we are arranging

In the example we had to arrange fork, knife and spoon. The same funda is used whether we had to arrange digits of a number, alphabets or people.

Forming a three-digit number using the digits 1, 2, 3 is same as arranging the objects 1, 2 and 3 in three specific positions viz. hundreds, tens and units.

Arranging the alphabets of CLAT in all possible ways is also the same as arranging four objects C, L, A and T in four positions.

Seating five people in a row for a photograph is also the same as arranging 5 objects, the five different people, in a row having 5 positions.

Thus, for Maths, it makes no difference if we are arranging knife, fork, spoon, etc or digits or alphabets or people. The funda remains the same whatever the context is. It's just that each of the context gives rise to different conditional arrangements. E.g. while arranging numbers we could have a condition that number has to be even or number has to be divisible by 3. While arranging people such conditions do not have any meaning. In the context of arranging people we can have conditions like two people want to sit together or they do not want to sit together.

In this case we had to arrange three objects in three places. What if the number of objects was more than the number of positions? We have already solved this situation when we were forming a three-digit number using the digits 1, 2, 3, 4 or 5, each digit used once.

It's obvious that not all the objects can be arranged. The first position can be filled in 5 ways, *for each of these 5 ways*, the second position *can be further filled in 4 ways. For each of the earlier ways* of filling the hundreds and tens position, the units position *can be further filled in 3 ways*. Thus, total number of ways of arranging is 5 × 4 × 3 = 60 ways. Thus the reasoning remains the same.

Defining ${}^{n}P_{r}$

As seen earlier,

the number of ways of arranging 3 objects out of 3 objects is 3 × 2 × 1

the number of ways of arranging 3 objects out of 5 objects is 5 × 4 × 3

By the same logic,

the number of ways of arranging 3 objects out of 11 objects would be 11 × 10 × 9

the number of ways of arranging 6 objects out of 9 objects would be 9 × 8 × 7 × 6 × 5 × 4

To generalise the above, the number of ways of arranging *r* objects in a row out of *n* distinct objects would be $\underbrace{n\times(n-1)\times(n-2)\times......}_{\text{total of } r \text{ factors}}$. This expression is denoted as ${}^{n}P_{r}$.

Definition: The number of ways of arranging r objects in a row out of n distinct objects is denoted as nPr and is given by the formula: ${}^{n}P_r = \frac{n!}{(n-r)!}$

This is the definition and to calculate the value one would have to do $\underbrace{n \times (n-1) \times (n-2) \times ......}_{total\ of\ r\ factors}$

A few permutations that should be known directly without using the formula are …

${}^{n}P_1 = n$ (Since we have to just choose 1 object, it could be any of the given n objects and can be done in n ways) ${}^{n}P_n = n!$ (The first position can be filled in n ways; second in $(n - 1)$ ways; third in $(n - 2)$ ways and the n^{th} position in just 1 way, the last object left.
Thus the number of ways is $n \times (n - 1) \times (n - 2) \times \ldots\ldots \times 1 = n!$

Arrangement of digits

Forming numbers, as seen above, is same as 'arranging digits' and the funda to be used in the basic Rule of *AND*. However while forming numbers, there can be many conditions like forming even numbers, forming numbers greater than a given number, forming numbers that are divisible by a given number etc. All the following examples show how to handle situations with such conditions……

E.g. 3: Using digits 1, 2, 3 and 4, how many 3 digit numbers can be formed if repetition of digits is not allowed?

While this question may appear very easy since we have already discussed this, do pay attention to the explanation in the box since it makes a few observations which will be useful to us later …

The number of ways in which the 1^{st} place, 2^{nd} place, 3^{rd} place can be filled is 4, 3, and 2 respectively. Thus the total possible numbers that can be formed is 4 × 3 × 2 = 24 different numbers can be formed. This is same as ${}^{4}P_3$, in notation.

E.g. 4: Using digits 1, 2, 3 and 4, how many 3 digit numbers can be formed if repetition of digits is allowed?

Since a digit can be re-used, each of the 3 positions can be filled in 4 ways and thus the total possible numbers that can be formed is 4 × 4 × 4 = 64.

In this example we could not use ${}^{n}P_r$ because repetitions are allowed.

E.g. 5: Using digits 1, 2, 3, 4, 5, 6 and 7, how many 5 digit EVEN numbers can be formed, without repeating any digit?

In this question there is a condition on the digits that can occupy the units position. In questions like these, when there is a condition on a certain position, we would NECESSARILY have to start by filling this position first i.e. satisfying the condition.

The units place could be filled in with only 2, 4 or 6 i.e. in 3 ways. Having filled the units place, four more places have to be filled and we have 6 digits remaining. This can be done in a total of 6 × 5 × 4 × 3 = 360 ways (in notation this will be ${}^{6}P_4$).

For each of the 3 ways of filling the units place, *there are further 360 ways* to fill the remaining 4 places. Thus the total number of 5 digit even numbers that can be formed is 3 × 360 = 1080.

E.g. 6: Using digits 0, 1, 2, 3, 4 and 5, how many four digit numbers can be formed using each digit only once?

When 0 (zero) is being used

Whenever 0 can be used and one has to form, say a four-digit number, please be careful that 0 cannot be placed in the leading position else it will not be a four-digit number.

Thus, while it appears that there is no condition given in the question, there is an implicit condition that 0 cannot appear in the leading position. And as learnt above, since there is a condition, we will have to start by satisfying this condition first. If we fill in other positions (apart from leading position in this case) then at end we would not have a fixed number of ways in which the leading position can be filled for all the earlier cases.

So that 0 does not appear in the leading position, we have to start by filling the thousands place. It can be filled only in 5 ways using any one of 1, 2, 3, 4 or 5. After filling this place, we are left with 5 digits (including zero) and three places to fill and this can be done in 5 × 4 × 3 ways (0 can appear in any of the positions). Thus total number of required numbers that can be formed is 5 × 5 × 4 × 3 = 300.

E.g. 7: Using digits 1, 2, 3, 4, 5, 6 and 7 (without repetition), find the number of distinct 4 digit numbers that can be formed for each of the following conditions:

1. greater than 3000

2. even

The first two cases should be very easy as they are exactly similar to earlier examples...

1. The thousand's digit can be filled in 5 ways (any one of 3, 4, 5, 6 or 7). Now one is left with 6 digits and has to arrange 3 of them. This can be done in ${}^6P_3$ = 6 × 5 × 4 ways. Thus a total of 5 × (6 × 5 × 4) = 600 such numbers can be formed.

2. The unit's digit can be filled in three ways (2, 4 or 6). Now we are left with 6 digits and have to arrange 3 of them. This can be done in ${}^6P_3$ = 6 × 5 × 4 ways. Thus a total of (6 × 5 × 4) × 3 = 360 such numbers can be formed.

Arranging people in a row

As stated already arranging people in a row is also very much like forming numbers because for Maths it hardly makes any difference if you are placing digits in places or persons in positions. The only difference being that instead of conditions like number being even or divisible by 3, we could have different conditions like two or more people wanting to be together or not wanting to be together or being in alternate position.

E.g. 8: In how many ways can 6 people be arranged in a row when two of them are adamant about standing at either extreme position?

As learnt earlier, we should satisfy the given conditions first. The two people, say *A* and *B*, can be placed at the extreme ends in two ways viz. *A* at the left end and *B* at right or *A* at the right end and *B* at left (the right end is different from the left end). Now we are left with 4 people and 4 places, which can be arranged in 4 × 3 × 2 × 1 = 24 ways. Thus the total number of ways of arranging is 2 × 24 = 48 ways.

Condition for persons to be together:

Let us understand this with an example: There are 7 people who have to be arranged in a row. If three of them want to be together, in how many ways can the arrangement be done?

Since the three persons cannot be separated, consider them as a one (imagine tying all three with a rope and consider this group as one person when placing them in the row). Now we have to arrange 4 + 1 i.e. 5 persons in a row which can be done in ${}^5P_5$ = 5! = 120 ways. But for *each of these 120 ways*, the three persons which are considered as one can be *further arranged among themselves*, giving rise to different arrangements. The three persons can arrange themselves in 3! = 6 ways. Thus for *each of the 120* arrangements, there are *further 6 different arrangements* and the total number of arrangement will be 120 × 6 = 720.

E.g. 9: There are 4 boys and 4 girls to be arranged in a row. If all the 4 girls have to be together, in how many ways can the arrangement be done?

Considering all the four girls as one unit, we have to arrange 5 units in 5 places and this can be done in 5! ways. Now one of the units is of 4 girls, who can be arranged amongst themselves in 4! ways. Thus the total number of ways of arranging is 5! × 4!.

E.g. 10: In a group of 10 people, 4 speak on French, 3 speak only Spanish and the rest speak both French and Spanish. In how many ways can the 10 people be arranged in a row such that all those who speak only French are together and so are all those who speak only Spanish.

Considering the 4 who speak only French as 1 unit and the 3 who speak only Spanish also as 1 unit, we have to arrange 1 + 1 + 3 (rest) = 5 objects in a row. This can be done in 5! ways.

Now for each of the above 5! arrangement, the four French speaking persons can be arranged among themselves in 4! ways. Thus, the total number of ways in which the 10 people can be arranged considering the 3 who speak only Spanish as 1 unit is 5! × 4!.

Again for each of above 5! × 4! arrangements, the three Spanish speaking persons can be arranged among themselves in 3! ways.

Thus the answer would be 5! × 4! × 3!.

Conditions for persons not being together

Again let's understand this case with an example: There are 6 boys and 4 girls to be arranged in a row. If no two girls should stand together, in how many ways can the arrangement be done?

In questions where no two units should be together, we should first arrange the *other* units and then place the units that should not be together, one each in the spaces between the other units arranged.

Since no two girls should be together, arrange the *others*, 6 boys in this case, first. The six boys can be arranged in 6! ways. Now there are seven places created between the boys and at either end, as shown below for two random cases of arranging the boys:

$\_\_\ B_1\ \_\_\ B_2\ \_\_\ B_3\ \_\_\ B_4\ \_\_\ B_5\ \_\_\ B_6\ \_\_$

$\_\_\ B_1\ \_\_\ B_5\ \_\_\ B_6\ \_\_\ B_3\ \_\_\ B_2\ \_\_\ B_4\ \_\_$

For <u>each</u> of these 6! ways ……

…… if not more than one girl is placed in each of the 7 spaces, it will ensure that no two girls will be together. But we have 7 spaces and only 5 girls. It just means that two of the spaces will be vacant. An empty space would mean the boys will be adjacent, which is acceptable because nothing is mentioned about boys being or not being together.

Thus, 5 girls are to be placed and there are 7 available spaces. This can be done in $^7P_5 = 7 \times 6 \times 5 \times 4$ ways. Alternately using rule of *AND*, the first girl can be placed in 7 ways, second girl can be placed in 6 ways, third girl can be placed in 5 ways and fourth girl in four ways.

Thus, the total number of ways of arranging is 6! × (7 × 6 × 5 × 4).

E.g 11: There are 4 boys and 4 girls to be arranged in a row. If no two girls should stand together, in how many ways can the arrangement be done?

The four boys can first be arranged in 4! ways. Now there will be 5 places for the 4 girls and they could be arranged in 5 × 4 × 3 × 2 ways. Thus the total number of arrangements possible is 4! × 5 × 4 × 3 × 2.

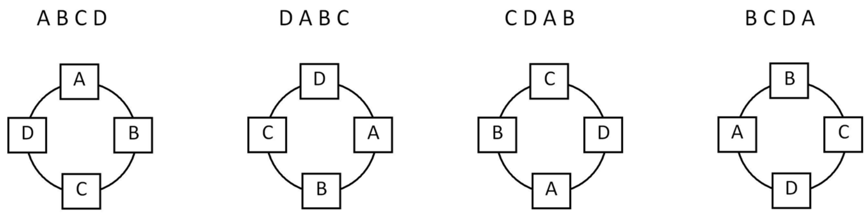
Conditions for persons to be alternate

E.g. 12: There are 4 boys and 4 girls to be arranged in a row. If girls and boys have to be alternate, in how many ways can the arrangement be done?

When girls and boys have to be alternate, it would just be either G B G B G B G B *or* B G B G B G B G. In *each* of these ways, there are 4 places for the boys and 4 places for the girls and thus they can be arranged in 4! × 4! in each of these. Thus, the total number of arrangements possible is 2 × 4! × 4!.

Circular Arrangement

Circular arrangement is distinct from linear arrangement. The following figure shows 4 distinct linear arrangements along with their corresponding arrangements in a circle.



Obviously all the four linear arrangements are different because different people occupy the extremes and there are numerous other differences related to right and left of each individual.

However in the case of circular arrangement, each of these 4 ways is exactly identical in every aspect *among themselves*. Not only does each individual have the same neighbour, but also the order of neighbours is the same. In all the four arrangements, B has A to *its* right and C to *its* left. (Consider them to be looking inside the circle).

The fact that the north position is taken by different individuals in the four possibilities is resorting to an "external reference" – north. In arrangements, we have to consider differences "among themselves". Thus, a circular arrangement is definitely distinct from a linear arrangement.

Let's take an example to understand the approach to solve a circular arrangement question.

In how many ways can four people be seated around a circular table with four equi-spaced chairs?

In fact whichever of the 4 chairs, the first occupant sits, all of them is the same. This is because there is no reference point as of now. Thus the first occupant can choose his position in only 1 way.

With the first person choosing his seat in 1 way, the second person now has three different ways (because a reference point exists i.e. the first person). The second person could sit opposite to the first person or to his left or to his right. Thus, the second person can choose his position in 3 ways; the third person can choose his chair in 2 ways; and the last person in 1 way. Thus all four of them can sit in 1 × 3 × 2 × 1 = 3! ways.

Consider 8 people to be seated across a circular table. For the 1st person though there are 8 chairs, the relative position of each chair is the same and hence he can choose his position in only 1 way. The moment he sits on any of the chairs, there is an internal reference (among themselves) created for the other people. The second person can choose his position in 7 ways, third can choose in 6 ways and so on. Thus, 8 people can be seated across a circular table in 1 × 7 × 6 × 5 × … × 1 = 7! ways.

In general, n people can be seated across a circular table in $(n - 1)!$ ways.

E.g. 13: 4 boys and 4 girls have to be seated around a circular table such that no two girls are adjacent to each other. In how many ways can they be seated?

Since it is a circular table and no two girls are adjacent to each other, they have to be on alternate chairs and the remaining set of alternate chairs will be occupied by the boys. Now the four boys can be seated in four alternate chairs in a circular arrangement in 3! ways. With the boys seated, there are 4 chairs for the girls and there is already a reference created because of the boys sitting. Thus the 4 girls can be seated in 4 chairs in 4! ways (and not 3!). Thus the total number of ways of seating is 3! × 4!.

Arranging Objects when few are identical like alphabets of a word

In how many ways can the letters of the word REVENUE be arranged among themselves?

This case is different from all earlier arrangements because one letter, E, appears multiple times in the word and hence we have to arrange seven letters of which 3 are identical. Thus $^7P_7$ will be wrong because that is only for arranging 7 distinct objects.

The total number of ways of arranging n objects of which m are identical is $\frac{n!}{m!}$

Thus, in this case, the total number of ways is $\frac{7!}{3!}$

The logic can extended as follows: If there n objects of which p objects are of one type, q objects of another type, r objects of yet another type and all others are distinct, the total number of ways of arranging all the objects is $\frac{n!}{p! \times q! \times r!}$

E.g. 14: In how many ways can the letters of the word "MISSISSIPPI" be arranged?

In the word MISSISSIPPI, there are 11 letters of which the alike ones are four S, four I, and two P. Thus they could be arranged in $\frac{11!}{4! \times 4! \times 2!}$.

E.g. 15: In how many ways can the letters of the word "OPPOSITION" be arranged so that the three O's and the S are together?

Since the three O and S has to be together, let's consider them as a group. Thus we have to arrange (OOOS), P, P, I, T, I, N. These are 7 objects of which two are P and two are I. They can be arranged in $\frac{7!}{2! \times 2!}$ ways.

Next, in the group of 3 O's and S, they can be arranged amongst themselves in $\frac{4!}{3!}$ ways.

Thus the total number of ways of arranging with the given condition is $\frac{7!}{2! \times 2!} \times \frac{4!}{3!}$.

Exercise

1. Using all digits from 0 to 9, how many 5 digit numbers can be formed? Repetition is allowed.

   a. 5^{10} b. 9^{10} c. 90,000 d. 5^5

2. How many 6 digit numbers can be formed using 1, 2, 3, 4, 5 and 6 such that all odd positions (from left end) are odd digits and all even positions are even digits? Repetition of digits is not allowed.

   a. 6! b. 3! × 3! c. 2 × 3! × 3! d. 18

3. Using digits 0, 1, 2, 3, 4, 5, 6 how many 5 digit natural numbers can be formed such that no digit is repeated in the number?

   a. 6 × 6! b. 7! / 2! c. 2160 d. 6 × 5!

4. How many 6 digit numbers can be formed using the digits 1, 2, 3, 4 and 5 (repetitions allowed) such that the number reads the same from left to right or from right to left (e.g. 134431)?

   a. 5^6 b. 5^3 c. 5^2 d. 5^{10}

5. For a group photograph, 4 girls and 5 boys have to be arranged in two rows, with all the girls sitting in the first row and all the boys standing in the second row. In how many ways can they be arranged?

   a. 4! × 5! b. 9! c. 2! × 4! × 5! d. None of these

6. In how many ways 8 friends be arranged in a row if three of the friends do not want to sit at the extremes?

   a. 8! b. 20 × 6! c. 3 × 6! d. 5 ×6!

7. 4 men and 4 women have to be seated in a row such that all the men are together and all the women are also together. In how many ways can they be seated?

   a. 4! × 4! b. 8! c. 2 × 4! × 4! d. 4!

8. 4 managers, 2 vice-presidents and 1 president have to be seated in a row for a meeting such that the two vice-presidents sit on either side of the president. In how many ways can they be seated?

   a. 7! b. 2 × 5! c. 2 × 4! × 5! d. 120

9. In how many ways can 8 cars be parked in 8 parking slots in a row such that there are exactly 4 cars between two specific cars?

   a. 2! × 3! × 4! b. 3 × 2! × 6! c. 4! × 4! d. 6! × 2!

10. In how many ways can the letters of the word INVESTOR be arranged such that no two vowels are together

    a. $\frac{6! \times 5!}{3!}$ b. $\frac{6! \times 5!}{3! \times 3!}$ c. $\frac{6!}{3! \times 3!}$ d. 5! × 6 × 5 × 4

11. In how many ways can 10 people be seated across a circular table if

    i. There are 10 identical chairs placed equally apart around the table

    ii. If there are 10 distinctly coloured chairs placed equally apart around the table.

    a. 10!, 10! b. 9!, 9! c. 9!, 10! d. 10!, 9!

12. 4 men and 4 women have to be seated in a circle such that all the men are together and all the women are also together. In how many ways can they be seated?

a. 4! × 4!   b. 3! × 4!   c. 7!   d. 8!

13. In how many ways can 6 Indians and 8 Americans sit across a circular table with 14 equi-spaced chairs such that no two Indians are sitting next to each other?

a. 7! × 6!   b. 5! ×8!   c. $7! \times {}^8P_6$ d. $5! \times {}^8P_7$

14. In how many ways can 6 couples be seated around a circular table such that each couple is sitting together?

a. $5! \times (2!)^6$ b. $6! \times (2!)^6$ c. $4! \times (2!)^5$ d. 12!/6!

15. In how many ways can the letters of the word "ABRACADABRA" be arranged?

a. 11!   b. $\frac{11!}{5!}$ c. $\frac{11!}{5!\times 2!\times 2!}$ d. $\frac{11!}{5!\times 4!}$

16. In how many ways can the letters of the word MATHEMATICS be arranged such that the two Ms are together, the two As are together and the two Ts are together?

a. 8!   b. $\frac{8!}{2!\times 2!\times 2!}$ c. $\frac{11!}{2!\times 2!\times 2!}$ d. None of these

17. How many numbers can be formed using each of the digits 1, 1, 2, 2, 2, 3 and 4?

a. 7!   b. $\frac{7!}{5!}$ c. $\frac{7!}{2!\times 3!}$ d. 7! × 2! × 3!

18. In how many ways can the letters of the word "ENGINEERING" be arranged such that all vowels are together?

a. $\frac{7!}{2!\times 3!}$ b. $\frac{5!\times 7!}{(2!\times 3!)^2}$ c. $\frac{11!}{5!\times 6!}$ d. $\frac{11!}{4!\times 7!}$

Selections (Combination)

So far our task was always to "arrange" objects i.e. to place them in a specific order among themselves.

Sometimes we would be interested in only "selecting" few objects out of the given objects. In this case we just need to "select" and we do not need to "arrange" them in an order. E.g. we need to select 4 students out of 15 students who will represent the college at a quiz or we need to form an academic committee of 3 professors from 10 professors. In this case, who is selected "first", who is selected "second" and so on does not matter. The words "first" and "second" implicitly implies an "ordering". What matters in the case of selection is only the composition of the final "group".

Comparison of Arrangement and Selection

Say we have to arrange 3 people out of A, B, C, D and E in a row.

We have seen that the arrangement A B C and A C B are distinct arrangements because of the relative positioning among themselves.

However if we had to select three people out of the 5 people, both of these "arrangements" would be the SAME "selection" viz. the group {A, B, C}.

In fact all the “arrangements” that are possible with A, B and C, in the case of “selection” would count as only 1 selection. Since A, B, C can be arranged among themselves in 3! i.e. 6 ways, all these 6 ways would be counted as only 1 if it were a case of selection and not arrangement.

A selection different from {A, B, C} would be only when the group composition is different, say {A, B, D} or {A, C, E} or {C, D, E}, etc. Also each of these groups would be counted as only 1 selection, irrespective of the ordering among the group members.

The above distinction should immediately make it clear that the number of selections will be lesser than the number of arrangements because quite a few of the arrangements are counted as just 1 selection. Conversely, with just 1 selection, we can order the group members among themselves and result in many more arrangements from the same selection.

See if you identify the difference between selection (combination) and arrangement (permutation) in similar looking contexts ....

Selection (Combination)	Arrangement (Permutation)
Selecting a team of 11 from 16 probable’s	Drawing a batting line-up of 11 from 16 probable’s
Selecting a committee of 3 from 10 members	Selecting a committee of a president, a vice-president and a treasurer from 10 members
Selecting 3 students out of 10 students who will receive scholarships of same value	Selecting 3 students out of 10 students who will receive three scholarships – one of Rs. 10,000, one of Rs. 5000 and one of Rs. 2000

Number of ways of selecting

Consider the same example as discussed in the above box: selecting 3 people out of A, B, C, D and E.

Rather than starting fresh, all over again, let us use the knowledge of arrangements that we have already learnt to find the number of ways in which we can select 3 objects out of 5 distinct objects.

Divide the task of arranging 3 people from 5 people into two sequential tasks ......

i. Select 3 people out of the 5 people

ii. Now arrange the 3 selected people among themselves.

Since we know that the number of ways in which 3 people selected can be arranged among themselves is 3!, hence,

(Number of selections) × 3! = (Number of arrangements)

$$\Rightarrow \text{Number of selection} = \frac{\text{Number of arrangements}}{3!}$$

Thus, to find the number of selections, we first find the number of arrangements and then use the fact that certain number of arrangements result in the same selection. We then arrive at the correct answer by dividing the number of arrangements by the number of arrangements resulting from each selection.

Defining ${}^{n}C_{r}$

The number of ways in which r objects can be selected from n distinct objects is denoted by ${}^{n}C_{r}$. and as explained earlier is same as $\frac{{}^{n}P_{r}}{r!}$

We already know that ${}^{n}P_{r} = \frac{n!}{(n-r)!}$ and thus, ${}^{n}C_{r} = \frac{n!}{(n-r)! \times r!}$

Obviously this is just the formula of ${}^{n}C_{r}$, but when we are finding the value of ${}^{n}C_{r}$, we would do the following calculation ....

$$^{n}C_{r} = \frac{\underbrace{n\times(n-1)\times(n-2)\times\ldots\ldots}_{r\ terms}}{r!}$$ i.e. $^{15}C_{4} = \frac{15\times14\times13\times12}{1\times2\times3\times4}$ and $^{10}C_{3} = \frac{10\times9\times8}{1\times2\times3}$

A few values that one should be very conversant with are…

$^{n}C_{1} = n$.

(And this is obviously so since if we have to select 1 out of n distinct objects, we can select the first or the second or the third or …… or the n^{th} i.e. in n different ways. Please remember that $^{n}P_{1}$ was also equal to n. This should also be obvious because when there is just 1 object, arranging l object has no meaning and thus will be same as selecting 1 object)

$^{n}C_{n} = 1$.

(Again this should be obvious since we have to select n out of n i.e. we have to select all of them and we do not have any choice to make the selection in any other manner)

In management entrance exams the questions asked on selection are relatively easier and stick to the basic format of selecting teams or committees.

E.g. 16: In how many ways can a committee of 7 members be chosen from 10 people?

The required answer is $^{10}C_{7} = \frac{10\times9\times8\times\cancel{7\times6\times5\times4}}{\cancel{7\times6\times5\times4}\times3\times2\times1} = \frac{10\times9\times8}{3\times2\times1} = 120$

$^{n}C_{r} = {}^{n}C_{n-r}$

As seen in the above expression, we had to write a lot of factors both in the numerator and the denominator, only to find that most of them cancel out.

Further if one is very observant, one can realise that after cancelling out the terms, we get a term which is exactly similar to the calculations of $^{n}C_{r}$ i.e. $\frac{10\times9\times8}{3\times2\times1} = {}^{10}C_{3}$

Thus, looking again at the calculation, $^{10}C_{7} = \frac{10\times9\times8\times\cancel{7\times6\times5\times4}}{\cancel{7\times6\times5\times4}\times3\times2\times1} = \frac{10\times9\times8}{3\times2\times1} = {}^{10}C_{3}$

In general $^{n}C_{r} = {}^{n}C_{n-r}$

Mathematical explanation:

$$^{n}C_{n-r} = \frac{n!}{(n-(n-r))!\times(n-r)!} = \frac{n!}{r!\times(n-r)!} = {}^{n}C_{r}$$

Logical explanation:

Consider the case of selecting r objects out n distinct objects. For every possible distinct selection, there would be a distinct left-over group of $(n-r)$ objects. Thus, the number of ways of selecting r objects would also be exactly equal to the number of ways of forming groups of $(n-r)$ objects from n objects.

Say we have to select 20 people from a group of 25 people. Rather than calling out the names of the 20 people who are selected, it is far easier to call out the names of 5 people who are not selected. Every possible ways of selecting 20 people has a unique group of 5 people not selected.

The above can be very useful to save our calculation when the number of objects to be selected is more than half the total objects.

E.g. 17: Out of 5 men and 6 women in how many ways can a committee of 2 men and 3 women be selected?

2 men can be selected out of 5 men in $^5C_2$ ways

3 women can be selected out of 6 women in $^6C_3$ ways.

Since we have to select 2 men *and* 3 women, it can be done in $^5C_2 \times {}^6C_3 = \frac{5 \times 4}{2!} \times \frac{6 \times 5 \times 4}{3!} =$

10 × 20 = 200 ways.

E.g. 18: Out of 5 men and 6 women in how many ways can a committee of 5 members be selected such that atleast 3 members are women?

Atleast 3 members are women implies that there could be 3 women *or* 4 women *or* 5 women in the committee. Thus, the required number of ways is

$$\underbrace{^6C_3 \times {}^5C_2}_{\text{3 women } and \text{ 2 men}} + \underbrace{^6C_4 \times {}^5C_1}_{\text{4 women } and \text{ 1 man}} + \underbrace{^6C_5}_{\text{5 women}}$$

$$= \frac{6 \times 5 \times 4}{3!} \times \frac{5 \times 4}{2!} + \frac{6 \times 5 \times 4 \times 3}{4!} \times \frac{5}{1!} + \frac{6 \times 5 \times 4 \times 3 \times 2}{5!}$$

= 20 × 10 + 15 × 5 + 6 = 281

However there are a few questions where you would need to identify the application of Selection and is not very apparent. These types of question could appear to be from any topic like reasoning (as seen in the following example) or geometry (as seen in few questions in the exercise)

E.g. 19: In a room there are 10 men and each man shakes hands with every other man present. How many hand-shakes take place?

A hand shake happens between every pair of person. Thus the number of hand-shakes is equal to the number of different pair of persons that can be formed from 10 people. The number of ways of choosing 2 persons out of 10 is $^{10}C_2 = \frac{10 \times 9}{2!}$ = 45. Thus 45 hand-shakes take place.

Alternate ways using reasoning and not P&C

Method 1: Each person is going to shake hands with 9 other people. And since there are 10 people, the total number of handshake will be 10 × 9. But then each hand-shake will be counted twice, once from each person's perspective and so the correct number of handshakes will be $\frac{10 \times 9}{2}$

Exercise

19. Each of three schools has sent a team of 4 for the competition. But only two from each school have to be selected. In how many ways can the selection be made?

    a. ${}^{34}C_2$ b. $4 \times {}^{4}C_2$ c. $({}^{4}C_2)^3$ d. None of these

20. In 16 probable's for the team there are 6 bowlers, 2 wicket-keepers and the rest are all batsmen. In how many ways can a team of 11 be selected such that 1 keeper and atleast 5 bowlers are chosen?

    a. ${}^{2}C_1 \times {}^{8}C_4 \times {}^{6}C_4$ b. ${}^{2}C_1 \times {}^{8}C_5 \times {}^{6}C_5$

    c. ${}^{2}C_1 \times ({}^{8}C_5 \times {}^{6}C_5 + {}^{8}C_4 \times {}^{6}C_6)$ d. None of these

21. In how many ways can a team of 5 be selected from 8 people such that either all three specific persons from the eight are selected or none of them are selected?

    a. 11 b. 9 c. 10 d. 8

22. In round-robin matches, each team plays every other team once. If there are 8 teams participating in the round-robin league matches, how many matches will be played?

    a. 28 b. 56 c. 64 d. 32

23. How many diagonals can be formed in a regular decagon? A decagon is a polygon with 10 vertices.

    a. 70 b. 35 c. 30 d. 50

24. On a plane there are 10 points, no three of which are collinear. How many triangles can be formed such that the vertices of the triangle are these points?

    a. 100 b. 10 c. 120 d. 90

25. Out of 5 woman and 6 men, in how many way can a committee of 3 be selected such that altleast one member of the committee is a woman.

    a. 175 b. 145 c. 165 d. 155

For questions 26 to 28: From 14 probable's, in which Sachin and Rahul are included, a team of 11 has to be selected.

26. In how many ways can the team be selected if both Sachin and Rahul have to be selected

    a. ${}^{12}C_9$ b. ${}^{14}C_{11} - {}^{11}C_2$ c. ${}^{14}C_{11}$ d. ${}^{14}C_{11} - {}^{12}C_9$

27. In how many ways can the team be selected if neither Sachin nor Rahul should be selected

    a. ${}^{14}C_{11} - {}^{12}C_9$ b. ${}^{12}C_{11}$ c. ${}^{14}C_{11} - {}^{12}C_2$ d. None of these

28. In how many ways can the team be selected such that if Sachin is selected, Rahul should not be selected and if Rahul is selected then Sachin should not be selected.

    a. $2 \times {}^{12}C_{10}$ b. ${}^{12}C_{10}$ c. ${}^{14}C_{10}$ d. $2 \times {}^{12}C_{10} + {}^{12}C_{11}$

Probability

Probability is the theory of quantifying the chance of an event occurring (or not occurring). While the theory is vast in itself, for the purpose of exam, very elementary questions are asked and hence we will stick to just the basics. All the questions of probability that are asked in entrance exams are based on discrete events and in the case of discrete events, probability is defined as:

Probability of an event occurring = $\dfrac{\text{number of favourable outcomes}}{\text{total number of possible outcomes}}$

By the formula it should be evident that probability can never be more than 1 because the number of favourable outcomes can never be more than the total number of outcomes.

Examples:

In a case of a coin being tossed, the total number of outcomes is just two viz. heads or tails turning up. If we want to find the probability of heads turning up on a toss, it is simply $\frac{1}{2}$.

When a regular dice is rolled, the total number of outcomes is 6 and thus the probability that a prime number is rolled will be greater than 2 will be $\frac{4}{6}$ since the favourable number of cases are rolling of 3 or 4 or 5 or 6.

When two cards are drawn out of a pack of cards, the total number of outcomes possible is ${}^{52}C_2$. If we want to find the probability of both the cards being face card, the two cards drawn must be from the 12 face cards and this can happen in ${}^{12}C_2$ ways. Thus the required probability is $\frac{{}^{12}C_2}{{}^{52}C_2}$

Tossing a coin, rolling a dice and drawing cards from a pack of cards are the most common scenarios in questions on probability.

E.g. 1: What is the probability that the sum of numbers turned, when two dice are rolled, is 10?

When two dice is rolled, there a total of 36 outcomes as shown below:

(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)
(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)	(2, 6)
(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)	(3, 6)
(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5)	(4, 6)
(5, 1)	(5, 2)	(5, 3)	(5, 4)	(5, 5)	(5, 6)
(6, 1)	(6, 2)	(6, 3)	(6, 4)	(6, 5)	(6, 6)

Please note that first dice turning 1 and second turning 2 is different from first dice turning 2 and second turning 1.

Required number of cases where the sum is 10 is (6, 4), (5, 5), (4, 6).

Thus the required probability is $\frac{3}{36}$.

E.g. 2: One card is drawn out from a pack of cards. What is the probability that the card drawn out is a King or a Red card?

One card can be drawn out in 52 ways.

While there are 26 Red cards and 4 Kings, the number of ways of drawing a King or a red card is not 26 + 4 i.e. 40. This is because 2 of the Kings are already counted in the Red card. Thus the number of cards from which a card favourable to the outcome can be drawn is 28. So the required probability = $\frac{28}{52} = \frac{7}{13}$.

Independent Events

Two events are said to be independent when the outcome of one of the event does not affect the outcome of the other event.

For two *independent* events, A and B, the probability that event A and B occurs is the product of the probabilities of event A and of event B i.e.

$p(A \text{ and } B) = p(A) \times p(B)$

For examples if a coin is tossed 5 times. The outcome on the second toss is not dependent on the outcome of the first toss. For that matter the outcome on any particular toss is not dependent on the outcome of the previous tosses. Thus each toss is an independent event. To find the probability of all five toss turning up Heads, we just need to multiply the probability of turning a head in each toss five times, i.e.

$$\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2} = \frac{1}{32}$$

Probability of a Union (Event *A or* Event *B*)

Let's say the probability of A solving a problem is $\frac{2}{3}$ and the probability of B solving the problem is $\frac{1}{2}$.

What is the probability that the problem is solved?

If we simply add the two probabilities, because we are finding the probability that A *or* B solves the problem, our answer will be $\frac{2}{3}+\frac{1}{2}=\frac{7}{6}$ which is greater than 1 and hence it is not possible.

Our answer is greater than 1 because *A* solving the problem is not exclusive to *B* solving the problem i.e. when we consider *A* solving the problem, we have made no mention of *B* and it could be possible that *B* has also solved the problem, which we independently add again. So the answer is greater than 1.

The way out is,

$p(A \text{ or } B) = p(A) + p(B) - p(A \text{ and } B)$

If they are independent events, we have already learnt that $p(A \text{ and } B) = p(A) \times p(B)$.

In this case, since the two guys solve problems independently, the required probability of the problem being solved is:

$p(A \text{ or } B \text{ solving}) = p(A \text{ solving}) + p(B \text{ solving}) - p(A \text{ solving}) \times p(B \text{ solving})$

$$= \frac{2}{3}+\frac{1}{2}-\frac{2}{3}\times\frac{1}{2} = \frac{4+3-2}{6} = \frac{5}{6}$$

The above question could also be solved by breaking the case of the problem being solved into cases that are exclusive to each other as follows:

A solved and B did not solve

A did not solve and B solved

Both of them solved.

Since these cases are exclusive the probabilities could directly be added and the answer can be found as $\frac{2}{3}\times\frac{1}{2}+\frac{1}{3}\times\frac{1}{2}+\frac{2}{3}\times\frac{1}{2}=\frac{2+1+2}{6}=\frac{5}{6}$

Complement of an Event:

If event *A* is a favourable event, the complement of *A* is defined as event *A not* occurring and the complement is denoted as A'.

Now it should be obvious that $p(A) + p(A') = 1$ because a event occurring and it not occurring would encompass all the possible outcomes.

This property of the probabilities of an event and its complement adding up to 1 can be used very effectively in certain cases as follows…

The case of 'Atleast 1'

When we are finding the probability of atleast one of the attempts turning a success, it is best solved by finding the probability of its complement and then subtracting it from 1 to find the answer. Because as just seen $p(A) = 1 - p(A')$.

The complement of 'atleast one attempt being a success' is 'none of the attempts being a success'

E.g. 3: A coin is tossed 6 times. What is the probability that atleast one toss results in a head?

One should immediately realize that this is a case of '*atleast 1*' and in breaking up this event into smaller exclusive events would be labourious as we could have 1 head or 2 head or 3 head or …so on.

The best strategy is to find the probability of the complement: 'probability that no toss results into a head'. Thus all toss should result into tail and

$p(\text{all 6 toss resulting in tails}) = \frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}=\frac{1}{64}$

$p(\text{atleast one toss being head}) = 1 - p(\text{all 6 toss resulting in tails}) = 1-\frac{1}{64}=\frac{63}{64}$

E.g. 4: In an oil-field, Reliance, Cairns and Shell are exploring for oil. The probability that Reliance strikes oil is $\frac{1}{3}$, that Cairns strikes oil is $\frac{1}{4}$ and that Shell strikes oil is $\frac{1}{5}$. Find the probability oil is found in the oil-field. Consider each company striking oil independent of each other.

Oil will be found if any of the company strikes oil i.e. atleast one of the company strikes oil. One can solve this by using set theory or by considering exclusive events. While there is a better method to solve this, first we shall solve this by both these methods just to reiterate the methods.

Using set theory, we want to find $p(R\cup C\cup S)$ and we can find this by using the formula,

$$p(R\cup C\cup S)=p(R)+p(C)+p(S)-\big(p(R\cap C)+p(C\cap S)+p(R\cap S)\big)+p(R\cap C\cap S)$$

$$= \frac{1}{3} + \frac{1}{4} + \frac{1}{5} - \left(\frac{1}{3} \times \frac{1}{4} + \frac{1}{4} \times \frac{1}{5} + \frac{1}{3} \times \frac{1}{5}\right) + \frac{1}{3} \times \frac{1}{4} \times \frac{1}{5}$$

$$= \frac{20 + 15 + 12 - (5 + 3 + 4) + 1}{3 \times 4 \times 5} = \frac{36}{3 \times 4 \times 5} = \frac{3}{5}$$

A slightly better way is to consider exclusive events…

Any of the following exclusive ways could be possible …

$R \not{C} \not{S} \quad \not{R} C \not{S} \quad \not{R} \not{C} S \quad R C \not{S} \quad R \not{C} S \quad \not{R} C S \quad R C S$

And the required probability is

$$\frac{1}{3} \times \frac{3}{4} \times \frac{4}{5} + \frac{2}{3} \times \frac{1}{4} \times \frac{4}{5} + \frac{2}{3} \times \frac{3}{4} \times \frac{1}{5} + \frac{1}{3} \times \frac{1}{4} \times \frac{4}{5} + \frac{1}{3} \times \frac{3}{4} \times \frac{1}{5} + \frac{2}{3} \times \frac{1}{4} \times \frac{1}{5} + \frac{1}{3} \times \frac{1}{4} \times \frac{1}{5}$$

$$= \frac{12 + 8 + 6 + 4 + 3 + 2 + 1}{3 \times 4 \times 5} = \frac{36}{3 \times 4 \times 5} = \frac{3}{5}$$

The above method of considering exclusive cases and then adding probabilities of them is a very useful technique.

However, for this question, the best way is to realize that the required probability involves "atleast 1" and thus finding the probability that none of the company strikes oil. This can occur in only one way that is when none of the company strikes oil. This is $\frac{2}{3} \times \frac{3}{4} \times \frac{4}{5} = \frac{2}{5}$. And the probability that atleast one company strikes oil is $1 - \frac{2}{5} = \frac{3}{5}$.

IMPORTANT NOTE:

Sometimes probability is expressed in terms of "odds in favour of" and "odds against". To convert such data into probability, understand the following…

When the odds in favour of an event is $a : b$, the probability of the event occurring is $\frac{a}{a+b}$.

When the odds against an event is $a : b$, the probability of the event occurring is $\frac{b}{a+b}$.

Exercise

1. In a simultaneous throw of two dice, what is the probability of getting a sum greater than 7?

   a. 5/12 b. 4/9 c. 1/2 d. 2/3

2. A bag contains 6 white and 4 black hats. Two hats are drawn simultaneously at random. What is the probability that both are of the same colour?

   a. 21/45 b. 1/3 c. 2/15 d. 24/45

3. From a pack of 52 cards, two cards are drawn simultaneously. What is the probability that the cards are of the same suit?

   a. 1/17 b. 4/17 c. 2/17 d. 3/17

4. In a class there are 15 boys and 10 girls. Three students are selected at random. What is the probability that 2 girls and 1 boy is selected?

   a. $\frac{^{10}C_2 \times {^{15}C_2}}{^{25}C_3}$ b. $\frac{^{10}C_2 \times {^{15}C_1}}{^{25}C_3}$ c. $\frac{^{10}C_2 \times {^{15}C_3}}{^{25}C_3}$ d. None of these

5. A box contains 20 bulbs, 12 are defective. Four bulbs are selected at random. What is the probability that atleast one of the four selected is defective?

   a. $1-\frac{^{8}C_4}{^{20}C_4}$ b. $\frac{^{12}C_4}{^{20}C_4}$ c. $1-\frac{^{12}C_4}{^{20}C_4}$ d. $\frac{^{8}C_4}{^{20}C_4}$

6. The probability that A solves the problem is $\frac{2}{3}$ and that B solves the problem is $\frac{3}{4}$. What is the probability that exactly one of them solves the problem?

   a. 5/12 b. 7/12 c. 1/2 d. 11/12

Set Theory

A set is any collection of objects satisfying a particular condition.

E.g. set of people drinking tea or set of people drinking coffee, set of students playing cricket, set of households subscribing to Times of India, etc. These are the sets that are gong to be most commonly used in problems.

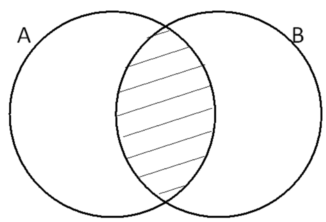
Each member of a set is called an element. We would be more importantly working with the number of elements in an particular set and if a set is denoted as A, the number of elements of it will be denoted as $n(A)$.

A set is usually denoted pictorially by a circle. All the elements of the set are enclosed in the circle.

Case of Two sets

Intersection of two sets

The intersection of two sets is the set of elements that are common to both sets. Pictorially the intersection of two sets is the shaded region as shown below:



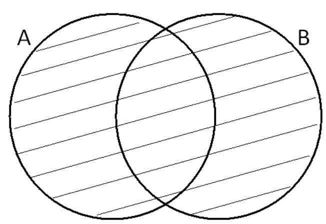
The intersection of the set of people who drink tea and the set of people who drink coffee is the set of people who drink both tea and coffee

Intersection of two sets A and B is denoted by $A \cap B$ in set terminology and its equivalence in language is set A and B.

Union of two sets

The union of two sets is the set of elements that belong to either of the two sets. Thus they include members that belong to only set A or only set B and also those that belong to both sets, the intersection.

Pictorially this is the shaded region:



The union of two sets A and B is denoted by $A \cup B$ in set terminology and its equivalence in language is set A or B.

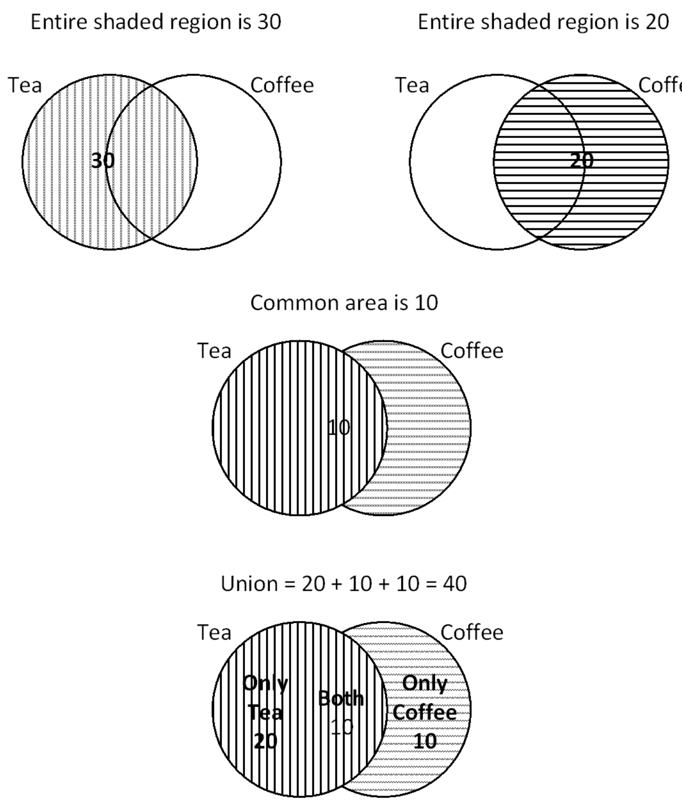
In finding the number of elements in $A \cup B$, we count all those present in the intersection of the two sets only once and not twice. In fact this is the funda on which all problems of set theory are based as explained below:

Let's say there are 30 people who drink tea and 20 people who drink coffee. Among them 10 are such that they drink both tea and coffee.

If the set of tea-drinkers is A and set of coffee drinkers is B, its obvious that $n(A \cap B)$ = 10 as directly given in the question.

But what is $n(A \cup B)$? Is it number of tea drinkers + number of coffee drinkers i.e. 30 + 20 = 50?

Not really, as you must have identified by now that when we add 30 + 20, the 10 people who are the intersection is counted twice. Thus for the correct number of tea or coffee drinkers we must subtract 10 from 50 as we have counted them twice, to get the correct answer as 40. If this is not clear by the explanation provided above, refer to the following diagram:



Thus the formula you would have to use in almost any problem on set theory based on two sets is:

$n(A \cup B) = n(A) + n(B) - n(A \cap B)$

Difference between Union and Sample Space

One last thing remains to be learnt: while most of the time our focus is on the union and intersection, there is one more set that is of importance, the set of elements that belong to neither of the two sets. This brings us to the funda of a Sample Space.

There is a difference between the following two sentences:

Statement 1: Out of 100 people who drink atleast one of tea or coffee, 70 drink tea and 50 drink coffee.

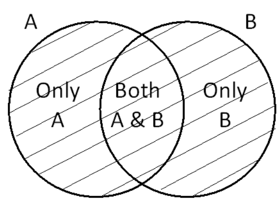
Statement 2: Out of 100 people, 70 drink tea and 50 drink coffee.

In statement 1, 100 is the union of the set of tea drinkers and coffee drinkers. Thus these 100 will be composed of those who only tea, those who drink only coffee or those who drink both.

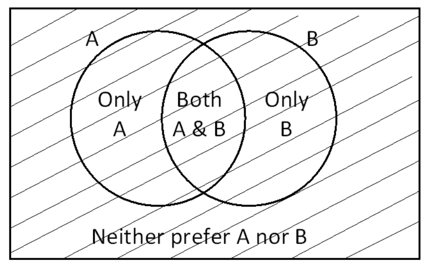
In statement 2, 100 is not the union but it is the total number of people or more appropriately the sample space. The difference is that few of these 100 people could not be drinking either of the two drinks. Say, 10 of them did not drink tea not coffee. Then the union of the tea drinkers and coffee drinkers would be 90 and not 100.

The difference between the two statements can also be understood from the following diagram:

Statement 1: The shaded area is 100
Or $A \cup B$ = 100



Statement 2: The shaded area is 100
$A \cup B$ could be less than 100



Thus in any question make sure that you do not confuse between the sample space and the union.

The union is usually expressed in any of the following language:

100 is the number of people drinking atleast one of the drinks $\Rightarrow A \cup B = 100$

100 is the number of people who drink tea or coffee $\Rightarrow A \cup B = 100$

Of 100 people, 10 do not drink tea nor coffee… $\Rightarrow A \cup B = 90$

E.g. 1: In a school of 60 students, each student has to choose atleast one elective out of French and German. 40 students chose French and 30 chose German. How many students chose both French and German?

Since each student has to choose atleast one of the elective, 60 is the union of set of students choosing French or German.

Hence 60 = 40 + 30 – $n(F \cap G)$

$n(F \cap G)$ = 70 – 60 = 10.

Thus 10 students chose both French and German

E.g. 2: In a club of 80 members, 45 play cricket, 30 play football and 10 play both cricket and football. How many do not play either cricket or football?

$n(C \cup F)$ = n(C) + n(F) – $n(C \cap F)$ = 45 + 30 – 10 = 65.

Thus remaining 80 – 65 = 15 do not play either of the game.

E.g. 3: In a survey done across 350 households, 270 subscribed to Times of India, 120 subscribed to Indian Express and 30 did not subscribe to either. Find the number of households who subscribed to only Times of India.

Out of 350 households, 30 did not subscribe to any newspaper. Thus the union of those subscribing to Times of India or Indian Express is 350 – 30 = 320.

Using $n(A \cup B) = n(A) + n(B) - n(A \cap B)$, we have

320 = 270 + 120 – $n(A \cap B)$

$n(A \cap B)$ = 390 – 320 = 70

Thus there are 70 households who subscribe to both Times of India and Indian Express. Removing these households from those subscribing Times of India will give us the number of households who subscribe to only Times of India.

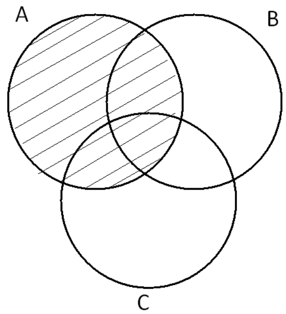
Thus those subscribing to only Times of India = 270 – 70 = 200.

Try solving all questions using the formula as explained above. Avoid using Venn diagrams as they consume more time and are used here just for explanation purpose and not for solving the questions.

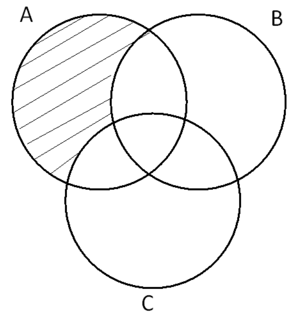
Case of Three sets

One of the first things to learn in the scenario of three sets is the language used. See the following figure very carefully to understand all the nuances of the language used…Let, A be the set of people playing cricket, B be the set of people playing football and C be the set of people playing hockey.

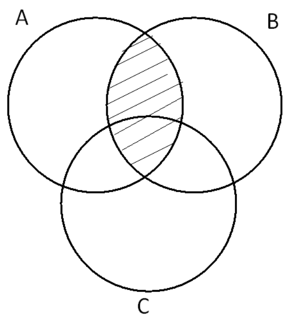
Set of people preferring A is the entire circle A.



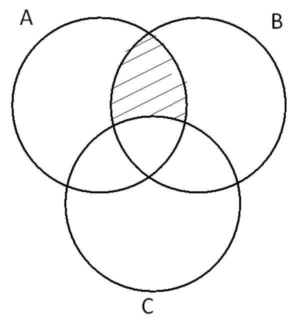
If we want to refer to the shaded area, we use ‘set of people preferring *only* A’



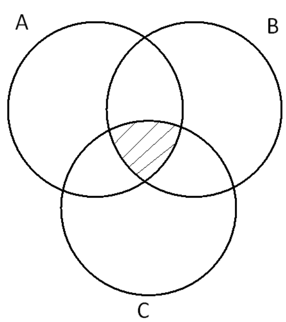
Set of people preferring A and B is the entire shaded area shown below, $A \cap B$



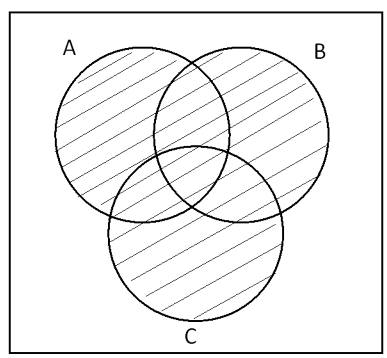
If we want to refer to the shaded area, we use ‘’set of people preferring *only* A and B’



Set of people preferring all three, A, B and C is the entire shaded area shown below, $A \cap B \cap C$



The ‘union $A \cup B \cup C$ ’or ‘people preferring A or B or C’ or ‘those preferring atleast one’ is the entire shaded area



The rectangle represents all the people i.e. the sample space. The area outside the shaded portion i.e. outside the union represents people who play none of the three sports.

In terms of set theory, the various notations are as explained below…

$A \cap B$: is the set of people who play cricket and football. Please note that this set includes people who play all three games. To exclude those who also play hockey, the language used is 'play *only* cricket and football'. Look at the first two pictures in the above diagram to understand the pictorial difference.

$B \cap C$: is the set of people who play football and hockey. Rest of the details is exactly as the above.

$A \cap C$: is the set of people who play cricket and hockey.

$A \cap B \cap C$: is the set of people who play all three games.

$A \cup B \cup C$: is the set of people who play atleast one game OR the set of people who play cricket or football or Hockey OR (the total number of people – the number of people who play none of cricket, football or hockey). Any of the above methods may be used to give the data regarding the union of the three sets.

The formula for the union in the case of three sets is:

$$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(A \cap C) + n(A \cap B \cap C)$$

E.g. 4: In a school having 150 students, 80 play cricket, 60 play football and 50 play hockey. 30 play cricket and football, 20 play football and hockey and 10 play cricket and hockey. If 5 play all three games, find the number of students who do not play any of the three games.

The number of students not playing any of the three games is given by 150 – union. And the union can be found out as (80 + 60 + 50) – (30 + 20 + 10) + 5 = 135.

Thus, the number of students not playing any game is 150 – 135 = 15.

Exercise

1. In a gathering it was found that 60% of the people present were more than 50 kgs in weight and 70% of the people were more than 5 feet tall. If 10% of the people satisfied neither of the two conditions, find the percentage of people who were both more than 50 kgs in weight and more than 5 feet tall.
   a. 30% b. 40% c. 50% d. 60%
2. In a class of 60 students, 40 students sing and 45 of them dance. If all students participate in atleast one of singing or dancing, find the number of students who participate in only one of the activity.
   a. 20 b. 35 c. 15 d. Cannot be determined
3. In a group, 40 people drink only tea, 30 drink only coffee and 90 drink atleast one of tea or coffee. If there were 10 who drink neither tea nor coffee, what percentage of the people present drink both tea and coffee?
   a. 15% b. 10% c. 20% d. Cannot be determined
4. In a group of 50 people, 30 do not drink tea and 20 do not drink coffee. If 15 of them drink both tea and coffee, how many drink neither tea not coffee?
   a. 15 b. 20 c. 30 d. Cannot be determined
5. In The World School, a student has the option to leave out Math or Science as a subject, but he cannot leave both. After 50 students dropped Math and 30 dropped Science, there were 20 students who opted studying both the subjects. Find the strength of the class.
   a. 80 b. 100 c. 70 d. 60
6. In a school, 120 students play cricket, 90 play football and 75 play hockey. Further, 50 play cricket and football, 40 play cricket and hockey and 25 play football and hockey. If the total number of students in the school are 200 and if 20 of them do not play any of the three games, find the number of students who play all three games.
   a. 20 b. 15 c. 10 d. 30

Heights and Distances

Common Right Angle Triangles and their trigonometric ratios

In questions on heights and distances, invariably, there would be a triangle with angles of 30°, 60°, 90° or 45°, 45°, 90°. One side of the triangle will be given and the other side of the triangle will be asked.

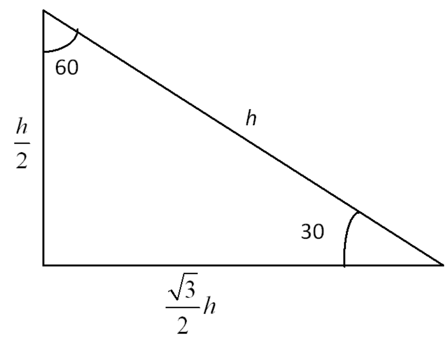
While these questions are based on the value of sine, cosine and tan of 30°, 45° and 60°, even if you are not aware of them, you can memorize the following relation between sides of the triangle and solve them. It is strongly advised that you attempt the questions on 'Heights and distance' if any of them come in the question paper as these are very easy once you memorize the following relation:

In a 30-60-90 triangle,

Side opposite to 30° is $\frac{1}{2}$ the hypotenuse

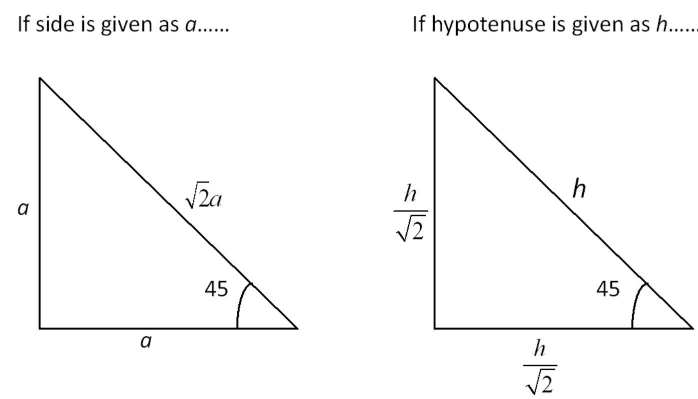
Side opposite to 60° is $\frac{\sqrt{3}}{2}$ times the hypotenuse.

Thus if the hypotenuse is considered to be h, you should just remember the following figure:



A 45-45-90 triangle is even easier…

Since it is an isosceles triangle, both the sides opposite to 45° are equal in length and are $\frac{1}{\sqrt{2}}$ times the hypotenuse. Alternately the hypotenuse is $\sqrt{2}$ times the side opposite to 45°.



Alternately, one should also be conversant with the following trigonometric ratios…

In a right angle triangle,

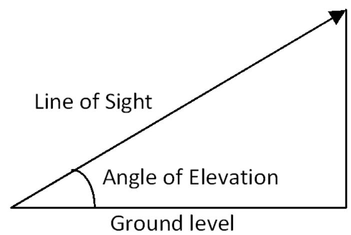
$\sin\theta = \frac{\text{opposite side}}{\text{hypotenuse}}$ $\quad\cos\theta = \frac{\text{adjacent side}}{\text{hypotenuse}}$ $\quad\tan\theta = \frac{\text{opposite side}}{\text{adjacent side}}$

And…

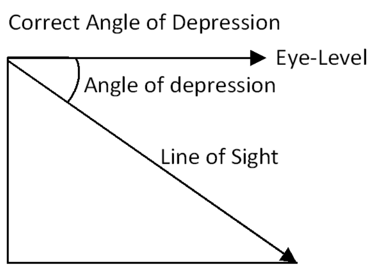
	30°	45°	60°
sin	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$
cos	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$
tan	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$

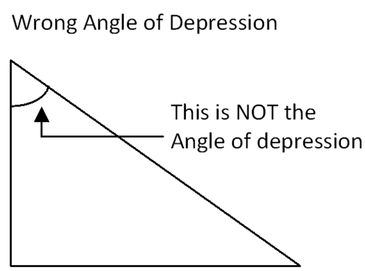
Angle of Elevation and angle of Depression

In all questions on Heights and Distances, the ground level is considered to be horizontal. When one has to look at a tower top or a pole top, one has to look upwards. The angle of elevation is the angle between the eye-level (ground-level in most of the problems) and the line of sight to the top of a tower, pole or mountain.

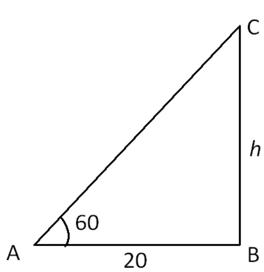


When one is standing on top of a building or mountain and viewing an object on the ground, one has to lower his line of sight from the horizontal line at eye-level. The angle between the horizontal line through the eye-level and the line of sight is the angle of depression. Many students make a mistake and consider the angle of depression as the angle between the line of sight and the vertical tower. This is wrong (as shown in the following diagram) and you should avoid committing this kind of mistake.



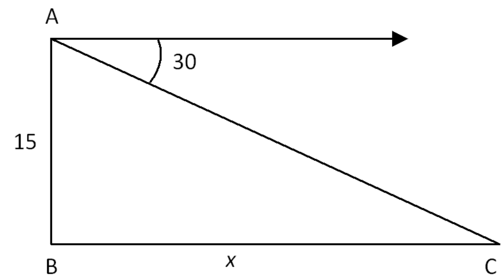


E.g. 1: The angle of elevation to the top of a tower from a point 20 ft from the base of the tower is 60°. Find the height of the tower?



Since AB is side opposite to the 30°, the hypotenuse will be twice the length of AB i.e. 40 feet. The height of the tower is the side opposite to 60° and hence will be $\frac{\sqrt{3}}{2}$ times the hypotenuse and will be $\frac{\sqrt{3}}{2} \times 40 = 20\sqrt{3}$ feet in this example.

E.g. 2: I am standing at the top of a vertical cliff that is 15 meters above ground level. The angle of depression to a point *x* meters from the base of the cliff is 30°. Find the value of *x*.

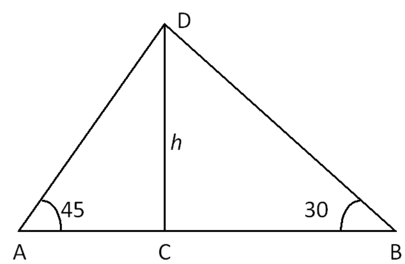


Since AB is opposite to 30°, it is half the hypotenuse.
Thus the hypotenuse, AC = 2 × 15 = 30

Since x is side opposite to 60°, it is $\frac{\sqrt{3}}{2}$ times the hypotenuse i.e. it is $15\sqrt{3}$

E.g. 3: Two points A and B are on either sides of a tower such that A, base of the tower and B are in a straight line. The angle of elevation from points A and B to the top of the tower is 45° and 30° respectively. If the distance between AB is 50 meters, find the height of the tower.

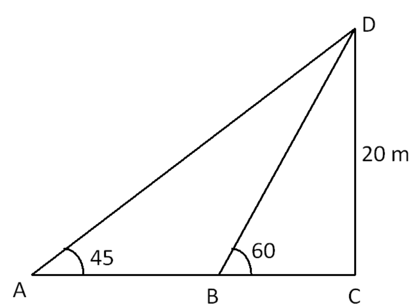
Let the height of the tower be h meters.



AC = h and CB = $\sqrt{3}h$.

Thus $h + \sqrt{3}h = 50 \Rightarrow h = \frac{50}{1+\sqrt{3}}$

E.g. 4: Two points A and B are on the same side of a tower such that A, B and the base of the tower are in a straight line. The angle of elevation from points A and B to the top of the tower is 45° and 60° respectively. If the height of the tower is 20 meters, find the distance AB.



In ΔBCD, we know CD = 20 meters. Since CD is side opposite to 60°, it is $\frac{\sqrt{3}}{2}$ time the hypotenuse. Thus hypotenuse BD = $\frac{2}{\sqrt{3}} \times 20 = \frac{40}{\sqrt{3}}$.

BC is side opposite to 30° and hence is half the hypotenuse and is of length $\frac{20}{\sqrt{3}}$ meters.

In ΔACD, since CD is 20 meters, so is AC, being a isosceles triangle.

AB = AC – BC = $20 - \frac{20}{\sqrt{3}} = \frac{20\left(\sqrt{3}-1\right)}{\sqrt{3}}$

Exercise

1. The angle of depression while viewing an object from a mountain top is 60°. If the height of the mountain is 75 meters, at what distance from the base of the mountain is the object?

   (1) $75\sqrt{3}$ (2) $50\sqrt{3}$ (3) $25\sqrt{3}$ (4) 150

2. A ladder of length 10 meters is leaning against a vertical wall such that the ladder makes and angle of 45° with the level ground. The base of the ladder slips away from the wall and now it is making an angle of 30° with the ground. By how many meters did the top of the ladder slide down along the vertical wall?

   (1) $5\left(2\sqrt{2}-1\right)$ (2) $5\left(\sqrt{2}-1\right)$ (3) $5\left(\sqrt{3}-1\right)$ (4) $5\left(2\sqrt{3}-1\right)$

3. The angle of elevation to the top of a tower is 45°. On walking 10 meters towards the tower, the angle of elevation becomes 60°. What is the height of the tower?

   (1) $5\left(\sqrt{3}+1\right)$ (2) $10\left(\sqrt{3}+1\right)$ (3) $\frac{10}{\left(\sqrt{3}+1\right)}$ (4) $15+5\sqrt{3}$

4. The angle of elevation to the top of a tower of height 50 meters is 45°. How many meters should one move backwards, so that the angle of elevation of the tower becomes 30°?

   (1) $50\left(\sqrt{3}-1\right)$ (2) $\frac{50}{\left(\sqrt{3}-1\right)}$ (3) $50-\frac{50}{\sqrt{3}}$ (4) $\frac{50}{\sqrt{3}}-10$

5. From a height of 20 meters above ground level, the angle of elevation to the top of a tower is 45° and from a height 30 meters above ground level, the angle of elevation to the top of the tower is 30°. Find the height of the tower from ground level.

   (1) $30+5\left(\sqrt{3}-1\right)$ (2) $20+5\left(\sqrt{3}-1\right)$ (3) $20+5\left(\sqrt{3}+1\right)$ (4) $30+5\left(\sqrt{3}+1\right)$

Calendars

Concept of Odd days

Questions on calendar ask us to identify which day of the week a given date falls on. To identify the day of the week we use the fact that the day of the week is the same after every 7 days i.e. a week. If it is a Sunday today, 7 days from now it will again be a Sunday. Also 14 days, or 21 days or 28 days or any multiple of 7 days from now, will be a Sunday.

Let's say today is Wednesday. What will be the day 45 days from today? Now we know that 42 days from today will again be a Wednesday. The extra days over complete number of week are called 'odd days'. In this example the number of odd days is 45 – 42 = 3. Thus the week-day 45 days from now will be 3 more days further than a Wednesday i.e. it will be a Saturday.

Thus, we would have to find the complete number of weeks from a reference point and the number of days extra than this, called 'odd days'.

The Gregorian Calendar

We follow the Gregorian calendar. The following facts about the Gregorian calendar are important to solve questions on calendar:

1. Years are divided as leap and non-leap years.
2. Non-leap years have 365 days i.e. 52 weeks and 1 odd day.

   Leap years have 366 days i.e. 52 weeks and 2 odd days.
3. A year is a leap year if the year is divisible by 4. Thus each of 2004, 2008, 2012, ... will be a leap year. But...

If a year is divisible by 100, for the year to be a leap year, it should be divisible by 400. Thus while 1200, 1600, 2000 are leap years, 1700, 1800, 1900, 2100 are not leap years (even though they are divisible by 4).

4. Each of Jan, Mar, May, July, Aug, Oct and Dec has 3 odd days.

   Each of Apr, Jun, Sep, Nov has 2 odd days.

   Feb of a leap year has 1 odd day and Feb of a non-leap year has 0 odd days.

Standard Questions on Calendars

Questions on calendar come in two types viz.

1. Finding the day of the week for a given date when a reference date and day is given.
2. Finding the day of the week for any given date without any reference.

Type 1: When a reference is given

In this case all you need to do is find the number of odd days from the date of reference to the date for which the day has to be found out. While doing so, be careful of the leap years that come in-between.

Also in all the working done below, we would INCLUDE the date for which the week-day is required to be found but we will EXCLUDE the date for which the week-day is given. Thus, if we get 1 odd day, it will be the next week-day in comparison to the given week-day. And if we have 2 odd days, the required week-day will be the given week-day + 2. And so on. If we get 0 odd days, the day of the week will be the same week-day as given.

E.g. 1: If 10th April 2007 is a Tuesday, what day of the week will 25th December 2010 fall on?

11th April 2007 to 10th April 2008 will have 2 odd days (a leap day of Feb 2008 will be in this period)

11th April 2008 to 10th April 2009 will have 1 odd day

11th April 2009 to 10th April 2010 will have 1 odd day

In the year 2010, rest of April (20 days), May, Jun, July, Aug, Sep, Oct, Nov and upto 25th Dec (inclusive) will have 20 + 3 + 2 + 3 + 3 + 2 + 3 + 2 + 25 = 63 i.e. 0 odd days.

Thus from 11th April 2007 to 25th Dec 2010 we would have 2 + 1 + 1 + 0 = 4 odd days.

Thus 25th December would be 4 days after a Tuesday i.e. it will be a Saturday.

In problems like this, one should ideally go directly from 11th April 2007 to 10th April 2010 and identify the odd days as 2 + 1 + 1 in one shot.

E.g. 2: If 14th August 1947 was a Monday, what day of the week was 26th January 1950?

From 15th August 1947 to 14th August 1949 we would have 2 + 1 = 3 odd days.

In the rest of days of Aug 1949 to 26th Jan 1950 we would have 17 + 2 + 3 + 2 + 3 + 26 = 53 i.e. 4 odd days.

Thus total odd days from 15th Aug 1947 to 26th Jan 1950 is 3 + 4 = 7 i.e. 0 odd day. Since 14th August is given to be a Monday, 26th Jan 1950 will also be a Monday.

E.g. 3: If 10th April 2007 is a Tuesday, what day of the week would 2nd October 2002 have been?

While the question asks us to go backwards, a safe strategy would be to go forward from 2nd Oct 2002 to 10th April 2007 and find the odd days in this period.

From 3rd Oct 2002 to 2nd October 2006 we would have 1 + 2 + 1 + 1 = 5 odd days.

Rest of Oct 2006 (29 days) and Nov 2006 to 10th April 2007 would give us 29 + 2 + 3 + 3 + 0 + 3 + 10 = 50 i.e. 1 odd days.

Thus from 3rd October 2002 to 10th April 2007 there are a total of 5 + 1 = 6 odd days.

Thus, 10th April 2007 will be 6 days of the week ahead than 2nd October 2002. And since this 6 days ahead is a Tuesday, 2nd October would be 6 days behind a Tuesday i.e. it will be a Wednesday.

Type 2: When no reference is given

In such questions you have to make do with the fact that the day on the beginning of the calendar was a Monday i.e. the date 1st Jan 0001 was a Monday.

Finding the odd number of days in a block of 100, 200, 300, 400 years

Strictly starting from the first day of the calendar,

The first 100 years (1st Jan 0001 to 31st Dec 0100) will have 24 leap years (0004, 0008, 0012, ……, 0096) and 76 non-leap years (note that the year 0100 is not a leap year).

Thus the first 100 years would have 24 × 2 + 76 odd days i.e. 124 odd days. These will amount to a total of 17 weeks and 5 odd days.

Thus first 100 years will have 5 odd days.

Similarly the next hundred years (1st Jan 0101 to 31st Dec 0200) would also have 5 odd days. Thus the first 200 years would have 5 + 5 = 10 i.e. 3 odd days.

The next (third) 100 years (1st Jan 0201 to 31st Dec 0300) would again have 5 odd days.

Thus the first 300 years will have 3 + 5 = 8 i.e. 1 odd day

But the fourth 100 years (1st Jan 0301 to 31st Dec 0400) would be different. It would have 25 leap years and 75 non-leap years. This is because the last year (0400) is a leap year whereas in the earlier cases the last years (0100, 0200, 0300) were not leap years.

Thus the fourth 100 years would have 2 × 25 + 75 = 125 i.e. 17 weeks and 6 odd days.

Thus the first 400 years would have 1 + 6 = 7 i.e. no odd days

This fact that first 400 years does not have any odd day is used as follows:

1st Jan of the years 0001, 0401, 0801, 1201, 1601, 2001, 2401 would all be Monday

Use this fact to come within 400 years of the given date.

Once you come to a date closest to the date given then we have to go forward in blocks of 100 years.

100 years would have 5 odd days

200 years would have 3 odd days

300 years would have 1 odd day.

Once we come even closer to the given date, we would have to go forward in blocks of 4 years.

Every block of 4 year has 3 non-leap years and 1 leap year. Thus it has 3 + 2 = 5 odd days.

After this we would have to go forward taking each year at a time.

If you follow the above strictly starting from the 1st of January 0001, (and INCLUDING 1st Jan 0001 and the given date as well) then the odd days so got would result in the day of week as follows:

0 odd days: Sunday   1 odd day: Monday   2 odd day: Tuesday   and so on till

6 odd day: Saturday.

Let's learn this with an example:

E.g. 4: Find the day of the week on 15th September 1995.

From the above stated fact, the first 1600 years would have 0 odd days.

From 1st Jan of the above mentioned years, the next,

100 years would have 5 odd days

200 years would have 5 + 5 = 10 i.e. 3 odd days

300 years would have 5 + 5 + 5 = 15 i.e. 1 odd day.

Use the above fact to come within 100 years of the given date.

From 1st Jan 1601 to 31st Dec 1900 is a period of 300 years and thus would have 1 odd day. You need not find the day of the week for the intermediate dates. You can simply go on collecting the odd days. Thus till 31st Dec 1900 we have collected 1 odd day.

Now, once you have reached within 100 years of the given date, move forward in blocks of 4 years till the time you reach within 4 years of the given date.

Each block of 4 years would have 5 odd days.

From 1st Jan 1901 to 31st Dec 1992 is a block of $\frac{92}{4} = 23$ four-year periods. Thus it will have

23 × 5 = 115 i.e. 3 odd days. Thus odd days collected so far are 1 + 3 = 4.

Once you reach within 4 years of the date given, we have inch forward on a year on year basis:

The year 1993 would have 1 odd day (cumulative odd days so far is 4 + 1 = 5)

The year 1994 would have 1 odd day (cumulative odd days so far is 5 + 1 = 6)

The year 1995 upto the date given i.e. 15th September (inclusive) would have 3 + 0 + 3 + 2 + 3 + 2 + 3 + 3 + 15 odd days for the months of Jan, Feb, Mar, April, May, Jun, July, Aug and September respectively. These total up to 34 i.e. 6 odd days.

Thus till 15th September, 1995 we collected a total of 6 + 6 = 12 i.e. 5 odd days.

1 odd day would imply that there is one more day than a complete week that started on Monday and ended on Sunday. Thus 1 odd day would imply the day is a Monday

Similarly 2, 3, 4, 5, 6, 0 odd days would imply the day is a Tuesday, Wednesday, Thursday, Friday, Saturday and Sunday respectively.

Thus 15th September, 1995 would be a Friday.

The above example was a probably the lengthiest one that you can expect. Most other dates would not be as lengthy…

E.g. 5: Find the day of the week on 5th May, 2007.

Every 400 years from the start of the calendar would yield 0 odd days. Thus, till 31st Dec 2000 we would get 0 odd days.

Now since the year asked is 2007, we do not have to go forward by any 100 years. So let's go forward by blocks of 4 years.

From 1st Jan 2001 to 31st Dec 2004, there would be 5 odd days.

The year 2005 would have 1 odd day.

The year 2006 would have 1 odd day.

The year 2007, till 5th May (inclusive) would have 3 + 0 + 3 + 2 + 5 = 13 i.e. 6 odd days.

Thus from start of the calendar to 5th May 2007, we have 5 + 1 + 1 + 6 = 13 i.e. 6 odd days.

Thus 5th May 2007 will be a Saturday.

Exercise

1. If 8th March, 2005 was a Wednesday, what was the day on 8th March 2004?

   (1) Tuesday (2) Wednesday (3) Thursday (4) Friday

2. If 10th January, 2009 will be a Saturday, what will be the day on 10th January, 2008?

   (1) Tuesday (2) Wednesday (3) Thursday (4) Friday

3. If CAT is always held on the third Sunday of November and in the year 2007 it was on the 18th of November, then on what date will it be held in 2009?

   (1) 15th (2) 16th (3) 19th (4) 20th

4. What was the day of the week on 16th July, 1776?

   (1) Monday (2) Tuesday (3) Saturday (4) Friday

5. What was the day of the week on 16th April, 2000?

   (1) Monday (2) Friday (3) Saturday (4) Sunday

Answer Key

Number Systems

1. d	2. a	3. c	4. d	5. b
6. c	7. b	8. c	9. d	10. c
11. b	12. d	13. c	14. b	15. c
16. b	17. b	18. c	19. b	20. c
21. d	22. a	23. a		

Assignment: Indices & Number Systems

1. c	2. c	3. d	4. d	5. b
6. a	7. b	8. a	9. a	10. b
11. c	12. a	13. c	14. b	15. d
16. c	17. c	18. a	19. a	20. d

Ratios:

1. b	2. b	3. b	4. a	5. b
6. d	7. c	8. d	9. a	10. b
11. b	12. a	13. a	14. b	15. c
16. a	17. c	18. b	19. d	20. b
21. c	22. d	23. d	24. a	25. c
26. b	27. b	28. d	29. b	30. d

Partnership

1. d	2. b	3. a	4. d	5. c

Proportion

1. d	2. d	3. a	4. b	5. c
6. b				

Variation – Chain Rule

1. d	2. d	3. c	4. c	5. d
6. b	7. a	8. a	9. c	10. c
11. b	12. a			

Direct and Inverse Variation

1. a	2. d	3. d	4. d	5. d

Assignment: Ratio Proportion Variation

1. b	2. a	3. d	4. b	5. a
6. c	7. b	8. b	9. d	10. c
11. a	12. a	13. b	14. d	15. c
16. c	17. d	18. c	19. c	20. b

Percentages

1. a	2. d	3. a	4. c	5. b
6. a	7. c	8. c	9. d	10. b
11. c	12. b	13. c	14. a	15. a
16. b	17. d	18. b	19. c	20. b
21. c	22. d	23. d	24. d	25. b
26. a	27. a	28. d	29. d	30. b
31. c	32. b	33. a	34. c	35. b

Profit Loss Discount

1. a	2. c	3. a	4. b	5. a
6. c	7. b	8. a	9. d	10. b
11. a	12. c	13. d	14. a	15. a
16. d	17. a	18. a	19. d	20. a

Simple & Compound Interest

1. b	2. b	3. d	4. c	5. c
6. d	7. d	8. b	9. d	10. a
11. d	12. c	13. b	14. a	15. c
16. b	17. c	18. d	19. a	20. c

Assignment: Percentages, PLD, SICI

1. b	2. c	3. c	4. d	5. b
6. c	7. b	8. d	9. c	10. b
11. a	12. d	13. b	14. c	15. a
16. b	17. b	18. b	19. a	20. b
21. b	22. d	23. b	24. b	25. d
26. c	27. d	28. b	29. a	30. a
31. b	32. c	33. c	34. b	35. d
36. d	37. d	38. d	39. b	40. a

Averages

1. d	2. d	3. b	4. b	5. b
6. d	7. a	8. b	9. a	10. c
11. b	12. b			

Weighted Averages

1. a	2. a	3. b	4. a	5. b
6. c	7. a	8. c	9. b	10. c
11. b	12. a	13. b	14. c	15. b

Assignment: Averages & Weighted Averages

1. d	2. c	3. a	4. b	5. b
6. a	7. d	8. b	9. b	10. d
11. b	12. d	13. c	14. b	15. b
16. a	17. d	18. c	19. b	20. b

Time Speed Distance

1. a	2. d	3. a	4. c	5. d
6. c	7. c	8. a	9. d	10. b
11. c	12. c	13. d	14. b	15. d
16. b	17. c	18. a	19. d	20. b
21. c	22. a	23. a	24. b	25. b
26. b	27. c	28. a	29. c	30. a
31. a	32. a	33. b	34. c	35. c

Time & Work

1. c	2. b	3. b	4. a	5. b
6. c	7. d	8. b	9. b	10. a
11. b	12. d			

Assignment: Time Speed Distance & Work

1. a	2. d	3. c	4. b	5. c
6. a	7. d	8. c	9. c	10. c
11. c	12. a	13. c	14. d	15. d
16. a	17. b	18. b	19. c	20. c

Algebra

1. b	2. a	3. c	4. a	5. c
6. d	7. a	8. b	9. c	10. b
11. a	12. a	13. a	14. c	15. c
16. a	17. c	18. b	19. c	20. c
21. c	22. b	23. a	24. a	25. b
26. c	27. b	28. c	29. a	30. b
31. a	32. b	33. c	34. b	35. c
36. b				

Mensuration

1. b	2. b	3. b	4. b	5. c
6. a	7. c	8. d	9. d	10. c
11. a	12. c	13. c	14. c	15. d

Assignment: Algebra & Mensuration

1. b	2. b	3. b	4. a	5. d
6. c	7. b	8. b	9. b	10. d
11. b	12. a	13. a	14. c	15. a
16. c	17. b	18. c	19. b	20. a

Permutation and Combination

1. c	2. b	3. c	4. b	5. a
6. b	7. c	8. b	9. b	10. d
11. c	12. b	13. c	14. a	15. c
16. a	17. c	18. b	19. c	20. c
21. a	22. a	23. b	24. c	25. b
26. a	27. b	28. d		

Probability

1. a	2. a	3. b	4. b	5. a
6. a				

Set Theory

1. b	2. b	3. c	4. a	5. b
6. c				

Heights & Distances

1. 3	2. 2	3. 4	4. 1	5. 4

Calenders

1. 1	2. 3	3. 1	4. 2	5. 4